1969

An algorithm for computing non-isomorphic semigroups of finite order.

Cullen, James Stephen

Monterey, California. U.S. Naval Postgraduate School

AN ALGORITHM FOR COMPUTING NON-ISOMORPHIC SEMIGROUPS OF FINITE ORDER

by

James Stephen Cullen

United States Naval Postgraduate School

THESIS

AN ALGORITHM FOR COMPUTING NON-ISOMORPHIC SEMIGROUPS OF FINITE ORDER

by

James Stephen Cullen

June 1969

An Algorithm for Computing Non-Isomorphic Semigroups

of Finite Order

by

James Stephen Cullen
Lieutenant (junior grade), United States Navy
B.S., United States Naval Academy, 1968

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE

from the

Naval Postgraduate School

ABSTRACT

In this paper an algorithm for computing semigroups of finite order is discussed. A computation procedure is developed to generate, for any specified finite order, all semigroups which are distinct up to isomorphism. Additional restrictions are also placed in the generating procedure to produce all groups of the given finite order. The algorithm was placed on the computer and the numerical results for orders one through four obtained.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

The author would like to thank Professor C. O. Wilde for his helpful suggestions in preparing this paper.

I. INTRODUCTION

In this paper we investigate the problem of computing all possible distinct algebraic systems of a certain type, namely the semigroup, with restrictions on the order only. A semigroup is an algebraic system which is closed and associative, and as such is the simplest algebraic system of significance in mathematics. More complex systems are determined by postulating additional properties, for example commutativity, many of which can easily be placed in a computation procedure. To illustrate this point a procedure was constructed to produce all groups in addition to all semigroups of the specified finite orders. In this paper we consider orders up to and including four only.

II. DEFINITIONS AND PRELIMINARY RESULTS

There are two basic alternatives for defining equivalence of semigroups. One approach is to identify two semigroups if the first is either isomorphic or anti-isomorphic to the second. The resulting collection of distinct semigroups is then described as a collection of non-equivalent semigroups. The alternate approach is to define one semigroup distinct from another if the first is not isomorphic to the second. The resulting collection is then described as a collection of semigroups which are distinct up to isomorphism. We use the latter approach in this paper and will explain the reasons for this choice later.

We begin by recalling the definitions of a binary operation on a set, of some properties a binary operation may possess, and of a semigroup itself.

Definition. A binary operation on a set S is a mapping from S S into S.

Definition. A binary operation Q is said to be associative if $Q(s_1,Q(s_2,s_3)) = Q(Q(s_1,s_2),s_3)$ for all $s_1,s_2,s_3 \in S$.

Definition. A binary operation Q is said to commutative if $Q(s_1,s_2) = Q(s_2,s_1)$ for all $s_1,s_2 \in S$.

Definition. A semigroup is a couple (S,Q) where S is a set on whose elements is defined an associative binary operation Q. The notation $Q(s_1,s_2) = s_3$ is somewhat cumbersome and will be used interchaneably with $(s_1 \cdot s_2) = s_3$ from this point on.

The order of a semigroup will mean the number of elements in the underlying set. For any given positive integer n there exists at least one semigroup of order n. For example, let $S = \{1,\dots,n\}$ and define

$$Q(i,j) = \begin{cases} i + j & \text{if } i + j \leq n \\ i + j - n & \text{if } i + j > n \end{cases}$$

for $i,j \in S$. This is the cyclic semigroup on n elements.

In this paper we compute the distinct (up to isomorphism) binary operations on a set of finite order satisfying the above conditions. One way to specify a binary operation on a finite set is by means of the multiplication table.

Example. Let $S = \{1,2,3\}$ and define the binary operation Q by the following table.

	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

The notation means that $Q(1,1) = 1$, $Q(1,2) = 2$, $Q(1,3) = 3$, $Q(2,1) = 2$, $Q(2,2) = 3$, and so on.

We use the following definitions as the basis for the construction of the isomorphism testing subroutine of the generation procedure.

Definition. Two semigroups (S_1,Q_1) and (S_2,Q_2) are called isomorphic if there exists a one-to-one mapping F of S_1 onto S_2 such that if $s_1,t_1 \in S_1$ and $F(s_1) = s_2$, $F(t_1) = t_2$ with $s_2,t_2 \in S_2$, then $F(Q_1(s_1,t_1)) = Q_2(s_2,t_2)$.

Definition. Two semigroups (S_1,Q_1) and (S_2,Q_2) are called anti-isomorphic if there exists a one-to-one mapping G of S_1 onto S_2 such that if $s_1,t_1 \in S_1$ and $G(s_1) = s_2$, $G(t_1) = t_2$ with $s_2,t_2 \in S_2$, then $G(Q_1(s_1,t_1)) = Q_2(t_2,s_2)$.

Given any semigroup (S,Q) we can in a natural way associate with it a semigroup (S^*,Q^*) defined by letting $S^* = S$, and for $s_1,s_2 \in S^*$ putting

$Q^*(s_1,s_2) = Q(s_2,s_1)$. If G is the identity mapping on S, then $G(Q(s_1,s_2)) = Q^*(s_2,s_1)$ and hence (S,Q) is anti-isomorphic to (S^*,Q^*).

If a semigroup (S,Q) is commutative, then it is both isomorphic and anti-isomorphic with (S^*,Q^*) since the identity map G is both an isomorphism and an anti-isomorphism. In the journal, Mathematical Algorithms, 1967, the editor remarked that the converse is also valid. However, the converse is not valid in general, as the following example illustrates.

Example. Let (S,Q) be defined by the table

	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	2	1

then (S^*,Q^*) is defined by this table

	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	1	1

where the second table was determined from the first by $Q^*(s_1,s_2) = Q(s_2,s_1)$ with the only differences in the multiplication tables being $Q^*(3,4) = Q(4,3)$ and $Q^*(4,3) = Q(3,4)$. The anti-isomorphism G is the identity mapping, while the mapping F defined by $F(1) = 1$, $F(2) = 2$, $F(3) = 4$, and $F(4) = 3$ is an isomorphism linking (S,Q) with (S^*,Q^*). It is interesting to note that orders two and three contain no non-commutative semigroup (S,Q) which is both isomorphic and anti-isomorphic to its (S^*,Q^*), but that order four contains six such semigroups. By computing non-equivalent semigroups the question whether or not a given semigroup is isomorphic to its anti-isomorphic image is left unanswered

unless the semigroup in question is commutative. In computing semigroups distinct up to isomorphism we avoid this difficulty.

III. ALGORITHM FOR COMPUTING SEMIGROUPS OF FINITE ORDER

Using the fact that the only condition on the algebraic system under consideration in addition to closure is associativity, we are able to build a systematic generating procedure. Noting that in any given multiplication table if $Q(i,j) = k$, then $Q(k,m) = Q(i,Q(j,m))$, we construct a procedure to complete any partially completed table. In essence, we utilize the associative law to complete the unfilled portion of the table. The first step consists of placing a few key values in a blank multiplication table. Then as much as possible of the table is completed by the application of the above equation.

<u>Example</u>. Let $S = \{1,2,3,4\}$ and let Q be partially defined by the following table.

	1	2	3	4
1	1	3		
2	2	1	3	
3		4		
4				

Then by applying the above associativity equation to the table we find $(3 \cdot 1) = ((1 \cdot 2) \cdot 1) = (1 \cdot (2 \cdot 1)) = (1 \cdot 2) = 3$ and $(1 \cdot 3) = (1 \cdot (1 \cdot 2)) = ((1 \cdot 1) \cdot 2) = (1 \cdot 2) = 3$ as well as $(1 \cdot 4) = 4$, $(2 \cdot 4) = 4$, $(4 \cdot 1) = 4$, and $(4 \cdot 2) = 3$. We also have $(4 \cdot 3) = (3 \cdot 3)$ and $(4 \cdot 4) = (3 \cdot 4)$. The following table results.

	1	2	3	4
1	1	3	3	4
2	2	1	3	4
3	3	4	x	y
4	4	3	x	y

Of course, in supplying the original six values a check must be made to determine whether or not they satisfy the associative law to make the

completion of the table worthwhile. The table must be checked again when completed since not all associativity conditions need necessarily be used in the completion of it.

The above example points out the difficulties that arise when a partially completed table has values which are not conducive toward further generation. When this happens additional values must be supplied to restart the generation procedure. The choosing of these additional values must follow a pattern and must exhaust all possibilities. In the above example the values 1,2,3, and 4 must be tested in both the x and y positions, which results in a total of 16 additional cases to be checked.

In the case of order four we set initial values in six positions. We chose the positions (1,1), (1,2), (2,1), and (2,2) to facilitate the generation procedure by filling a fourth of the blank multiplication table, while we picked the two additional positions (2,3) and (3,2) to be used as launching points for the completion of the table. We chose an initial six positions for two reasons. First, since every finite semigroup has an idempotent element,[1] then every finite semigroup is isomorphic to some semigroup with the element one (1) in the position (1,1). If in a semigroup of order n the element i is idempotent, then define an isomorphism F such that $F(i) = 1$, $F(1) = i$, and $F(k) = k$ for $k = 2,\ldots,i - 1, i + 1,\ldots,n$. The second reason is that the remaining five positions have to be filled in a manner that exhausts all possible combinations of the values one (1) through four (4). The number of initial cases, ranging from the values 1,1,1,1,1 to the values

---

[1] Clifford, A. H. and Preston, G. B., The Algebraic Theory of Semigroups, v. 1, p. 20, American Mathematical Society, 1961.

4,4,4,4,4 in the five positions, is 4^5, or 1024. The addition of any more initial positions would increase this number by a factor of four for each added position.

We developed this generating procedure because of time limitations on the use of the computer. For orders two and three the number of positions set with initial values were four and nine respectively. We exhaustively checked all possible combinations in these two orders since the number of cases to be checked was low. For order two there were only 2^4 cases while for order three there were only 3^9 cases. However, for order four there are 4^{16}, or over four billion, cases to be examined, which proved to be much too time consuming to allow the exhaustive procedure.

IV. IMPLEMENTATION OF ALGORITHM

We divided the algorithm into three basic parts in order to place it on the computer using the Fortran programming system. The first and primary part consists of producing the completed multiplication tables by application of the associative equation, $Q(k,m) = Q(i,Q(j,m))$ when $Q(i,j) = k$, to the initial values of the partially completed tables. The initial values are supplied by use of nested "DO loops." The number of nested "DO loops" used were four, nine, and five for orders two, three, and four respectively. Hence, for orders two and three entire multiplication tables were set with initial values and the algorithm degenerated into an exhaustive test of every possible multiplication table.

For order four we found that seven nested "DO loops" would be the maximum number for practical purposes, that is, any more would result in too much time consumption. We decided on five for the reasons stated before. After the initial values are supplied we check for violations of the associative law. If there are none the computer then applies the associative equation to the initial values and then to generated values until the generation procedure ceases. At this point the number of blanks remaining in the multiplication table determines to which further generation subroutine the computer switches. Once the multiplication table is completed we are finished with the first part of the algorithm.

The second part of the algorithm consists of the main associativity test. We check the entire multiplication table since the generation procedure does not necessarily use every associativity condition. In this associativity test as well as in previous ones we make use of the "LOGICAL IF" statement and self-subscripting capabilities of the Fortran language.

Example. IF (I(KK,I(LL,MM)).NE.I(I(KK,LL),MM)) GO TO 100

Once a multiplication table passes the second part it is given a number and recognized as the representation of a semigroup.

In the third part of the algorithm we take these multiplication tables and determine those which are not isomorphic to any of the others. This select group then represents a collection of semigroups which are distinct up to isomorphism.

As we mentioned before, additional subroutines were added to produce all groups as well as semigroups of the specified finite orders. These subroutines follow the third part of the algorithm.

We include the programs used and the output obtained in the latter part of this paper.

TABLE I

ORDER	1	2	3	4
Number of semigroups distinct up to isomorphism	1	5	24	188
Number of commutative semigroups	1	3	12	58
Number of non-commutative isomorphic anti-isomorphic semigroups	0	0	0	6
Number of groups distinct up to isomorphism	1	1	1	2

```
SEMIGROUPS OF ORDER TWO DISTINCT UP TO ISOMORPHISM


  1     1
  1     1
         SEMIGROUP  NUMBER  IS      1
         SEMIGROUP  IS  COMMUTATIVE


  1     1
  1     2
         SEMIGROUP  NUMBER  IS      2
         SEMIGROUP  IS  COMMUTATIVE
         SEMIGROUP  HAS  LEFT  IDENTITY  X = 2  SUCH  THAT
            XY=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 2  SUCH  THAT
            YZ=Y
         SEMIGROUP  HAS  IDENTITY


  1     1
  2     2
         SEMIGROUP  NUMBER  IS      3
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 1  SUCH  THAT
            YZ=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 2  SUCH  THAT
            YZ=Y


  1     2
  1     2
         SEMIGROUP  NUMBER  IS      4
         SEMIGROUP  HAS  LEFT  IDENTITY  X = 1  SUCH  THAT
            XY=Y
         SEMIGROUP  HAS  LEFT  IDENTITY  X = 2  SUCH  THAT
            XY=Y


  1     2
  2     1
         SEMIGROUP  NUMBER  IS      5
         SEMIGROUP  IS  COMMUTATIVE
         SEMIGROUP  HAS  LEFT  IDENTITY  X = 1  SUCH  THAT
            XY=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 1  SUCH  THAT
            YZ=Y
         SEMIGROUP  HAS  IDENTITY
         SEMIGROUP  IS  A  GROUP
```

SEMIGROUPS OF ORDER THREE DISTINCT UP TO ISOMORPHISM

```
1     1     1

1     1     1

1     1     1
   SEMIGROUP   NUMBER     1
   SEMIGROUP   IS  COMMUTATIVE



1     1     1

1     1     1

1     1     2
   SEMIGROUP   NUMBER     2
   SEMIGROUP   HAS  A  SUBSEMIGROUP  OF  ORDER  TWO
        (1,2)
   SEMIGROUP   IS  COMMUTATIVE



1     1     1

1     1     1

1     1     3
   SEMIGROUP   NUMBER     3
   SEMIGROUP   HAS  A  SUBSEMIGROUP  OF  ORDER  TWO
        (1,2)
   SEMIGROUP   HAS  A  SUBSEMIGROUP  OF  ORDER  TWO
        (1,3)
   SEMIGROUP   IS  COMMUTATIVE



1     1     1

1     1     1

1     2     3
   SEMIGROUP   NUMBER     4
   SEMIGROUP   HAS  A  SUBSEMIGROUP  OF  ORDER  TWO
        (1,2)
   SEMIGROUP   HAS  A  SUBSEMIGROUP  OF  ORDER  TWO
        (1,3)
   SEMIGROUP   HAS  LEFT  IDENTITY  X = 3
          SUCH  THAT  XY=Y
```

```
1       1       1

1       1       1

3       3       3
         SEMIGROUP   NUMBER     5
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,3)


1       1       1

1       1       2

1       1       3
         SEMIGROUP   NUMBER     6
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,3)
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 3
                SUCH  THAT  YZ=Y


1       1       1

1       1       2

1       2       3
         SEMIGROUP   NUMBER     7
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,3)
         SEMIGROUP   IS  COMMUTATIVE
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 3
                SUCH  THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 3
                SUCH  THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       1       1

1       2       1

1       1       3
         SEMIGROUP   NUMBER     8
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
              (1,3)
         SEMIGROUP   IS  COMMUTATIVE
```

```
1      1      1

1      2      1

3      3      3
        SEMIGROUP   NUMBER     9
        SEMIGROUP   HAS  A   SUBSEMIGROUP  CF  CRDER  TWO
            (1,2)
        SEMIGROUP   HAS  A   SUBSEMIGROUP  CF  CRDER  TWO
            (1,3)
        SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y


1      1      1

1      2      2

1      2      2
        SEMIGROUP   NUMBER    10
        SEMIGROUP   HAS  A   SUBSEMIGROUP  CF  CRDER  TWO
            (1,2)
        SEMIGROUP   HAS  A   SUBSEMIGROUP  CF  CRDER  TWO
            (2,3)
        SEMIGROUP   IS  CCMMLTATIVE


1      1      1

1      2      2

1      2      3
        SEMIGRCUP   NUMBER    11
        SEMIGRCUP   HAS  A   SUBSEMIGROUP  CF  CRDER  TWO
            (1,2)
        SEMIGROUP   HAS  A   SUBSEMIGROUP  CF  CRDER  TWO
            (1,3)
        SEMIGROUP   HAS  A   SUBSEMIGROUP  CF  CRDER  TWO
            (2,3)
        SEMIGROUP   IS  CCMMLTATIVE
        SEMIGRCUP   HAS  LEFT  IDENTITY  X = 3
              SUCH  THAT  XY=Y
        SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 3
              SUCH  THAT  YZ=Y
        SEMIGROUP   HAS  IDENTITY


1      1      1

1      2      2

1      3      3
        SEMIGROUP   NUMBER    12
        SEMIGROUP   HAS  A   SUBSEMIGROUP  OF  ORDER  TWO
            (1,2)
        SEMIGROUP   HAS  A   SUBSEMIGROUP  OF  CRDER  TWO
            (1,3)
        SEMIGROUP   HAS  A   SUBSEMIGRCUP  CF  CRDER  TWO
            (2,3)
        SEMIGRCUP   HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y
        SEMIGRCUP   HAS  RIGHT  IDENTITY  Z = 3
              SUCH  THAT  YZ=Y
```

```
1       1       1

1       2       3

1       2       3
         SEMIGROUP   NUMBER    13
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (1,3)
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (2,3)
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 2
                 SUCH  THAT  XY=Y
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 3
                 SUCH  THAT  XY=Y


1       1       1

1       2       3

1       3       2
         SEMIGROUP   NUMBER    14
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (2,3)
         SEMIGROUP   IS  COMMUTATIVE
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 2
                 SUCH  THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 2
                 SUCH  THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       1       1

1       2       3

3       3       3
         SEMIGROUP   NUMBER    15
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (1,3)
         SEMIGROUP   HAS  A   SUBSEMIGROUP  OF   ORDER  TWO
             (2,3)
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 2
                 SUCH  THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 2
                 SUCH  THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY
```

```
1        1        1

2        2        2

3        3        3
          SEMIGRCUP   NUMBER    16
          SEMIGRCUP   HAS  A   SUBSEMIGROUP   OF   CRDER   TWC
               (1,2)
          SEMIGRCUP   HAS  A   SUBSEMIGROUP   CF   CRDER   TWC
               (1,3)
          SEMIGRCUP   HAS  A   SUBSEMIGRCUP   CF   CRDER   TWC
               (2,3)
          SEMIGRCUP   HAS  RIGHT   IDENTITY   Z = 1
                 SUCH   THAT   YZ=Y
          SEMIGRCUP   HAS  RIGHT   IDENTITY   Z = 2
                 SUCH   THAT   YZ=Y
          SEMIGRCUP   HAS  RIGHT   IDENTITY   Z = 3
                 SUCH   THAT   YZ=Y


1        1        3

1        1        3

1        1        3
          SEMIGRCUP   NUMBER    17
          SEMIGRCUP   HAS  A   SUBSEMIGROUP   CF   ORDER   TWO
               (1,2)
          SEMIGRCUP   HAS  A   SUBSEMIGRCUP   CF   CRDER   TWO
               (1,3)


1        1        3

1        1        3

3        3        1
          SEMIGRCUP   NUMBER    18
          SEMIGRCUP   HAS  A   SUBSEMIGRCUP   CF   ORDER   TWO
               (1,2)
          SEMIGRCUP   HAS  A   SUBSEMIGROUP   OF   CRDER   TWO
               (1,3)
          SEMIGRCUP   IS   CCMMLTATIVE


1        1        3

1        2        3

1        1        3
          SEMIGROUP   NUMBER    19
          SEMIGRCUP   HAS  A   SUBSEMIGROUP   CF   CRDER   TWO
               (1,2)
          SEMIGRCUP   HAS  A   SUBSEMIGRCUP   CF   CRDER   TWO
               (1,3)
          SEMIGRCUP   HAS  LEFT   IDENTITY   X = 2
                 SUCH   THAT   XY=Y
```

```
1      1      3

1      2      3

1      3      3
        SEMIGROUP   NUMBER    20
        SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
            (1,2)
        SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
            (1,3)
        SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
            (2,3)
        SEMIGROUP   HAS  LEFT  IDENTITY  X = 2
              SUCH  THAT  XY=Y
        SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y
        SEMIGROUP   HAS  IDENTITY


1      1      3

1      2      3

3      3      1
        SEMIGROUP   NUMBER    21
        SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
            (1,2)
        SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
            (1,3)
        SEMIGROUP   IS  COMMUTATIVE
        SEMIGROUP   HAS  LEFT  IDENTITY  X = 2
              SUCH  THAT  XY=Y
        SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y
        SEMIGROUP   HAS  IDENTITY


1      2      2

2      1      1

2      1      1
        SEMIGROUP   NUMBER    22
        SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
            (1,2)
        SEMIGROUP   IS  COMMUTATIVE
```

```
1       2       3

1       2       3

1       2       3
         SEMIGROUP   NUMBER    23
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
             (1,2)
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
             (1,3)
         SEMIGROUP   HAS  A   SUBSEMIGROUP   OF   ORDER   TWO
             (2,3)
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 1
               SUCH  THAT   XY=Y
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
               SUCH  THAT   XY=Y
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
               SUCH  THAT   XY=Y


1       2       3

2       3       1

3       1       2
         SEMIGROUP   NUMBER    24
         SEMIGROUP   IS   COMMUTATIVE
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 1
               SUCH  THAT   XY=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 1
               SUCH  THAT   YZ=Y
         SEMIGROUP   HAS  IDENTITY
         SEMIGROUP   IS   A   GROUP
```

SEMIGROUPS OF ORDER FOUR DISTINCT UP TO ISOMORPHISM

```
1       1       1       1
1       1       1       1
1       1       1       1
1       1       1       1
     SEMIGROUP  NUMBER        1
     SEMIGROUP  IS  COMMUTATIVE

1       1       1       1
1       1       1       1
1       1       1       1
1       1       1       2
     SEMIGROUP  NUMBER        2
     SEMIGROUP  IS  COMMUTATIVE

1       1       1       1
1       1       1       1
1       1       1       1
1       1       1       4
     SEMIGROUP  NUMBER        3
     SEMIGROUP  IS  COMMUTATIVE

1       1       1       1
1       1       1       1
1       1       1       1
1       1       2       1
     SEMIGROUP  NUMBER        4

1       1       1       1
1       1       1       1
1       1       1       1
1       1       2       2
     SEMIGROUP  NUMBER        5
```

```
1       1       1       1

1       1       1       1

1       1       1       1

1       1       3       4
         SEMIGROUP   NUMBER        6


1       1       1       1

1       1       1       1

1       1       1       1

1       2       2       4
         SEMIGROUP   NUMBER        7


1       1       1       1

1       1       1       1

1       1       1       1

1       2       3       4
         SEMIGROUP   NUMBER        8
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 4
                 SUCH   THAT   XY=Y


1       1       1       1

1       1       1       1

1       1       1       1

4       4       4       4
         SEMIGROUP   NUMBER        9


1       1       1       1

1       1       1       1

1       1       1       2

1       1       1       2
         SEMIGROUP   NUMBER       10


1       1       1       1

1       1       1       1

1       1       1       2

1       1       2       1
         SEMIGROUP   NUMBER       11
         SEMIGROUP   IS   COMMUTATIVE
```

```
1       1       1       1

1       1       1       1

1       1       1       2

1       1       2       2
         SEMIGROUP   NUMBER       12
         SEMIGROUP   IS   COMMUTATIVE

1       1       1       1

1       1       1       1

1       1       1       2

1       1       2       3
         SEMIGROUP   NUMBER       13
         SEMIGROUP   IS   COMMUTATIVE

1       1       1       1

1       1       1       1

1       1       1       3

1       1       1       4
         SEMIGROUP   NUMBER       14

1       1       1       1

1       1       1       1

1       1       1       3

1       1       3       4
         SEMIGROUP   NUMBER       15
         SEMIGROUP   IS   COMMUTATIVE

1       1       1       1

1       1       1       1

1       1       1       3

1       2       1       4
         SEMIGROUP   NUMBER       16

1       1       1       1

1       1       1       1

1       1       1       3

1       2       3       4
         SEMIGROUP   NUMBER       17
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 4
               SUCH   THAT   XY=Y
```

```
1     1     1     1

1     1     1     2

1     1     1     2

1     1     1     4
       SEMIGROUP   NUMBER      18


1     1     1     1

1     1     1     2

1     1     1     2

1     2     2     4
       SEMIGROUP   NUMBER      19
       SEMIGROUP   IS   COMMUTATIVE


1     1     1     1

1     1     1     2

1     1     1     2

1     2     3     4
       SEMIGROUP   NUMBER      20
       SEMIGROUP   HAS   LEFT   IDENTITY   X = 4
             SUCH   THAT   XY=Y


1     1     1     1

1     1     1     2

1     1     1     3

1     1     1     4
       SEMIGROUP   NUMBER      21
       SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
             SUCH   THAT   YZ=Y


1     1     1     1

1     1     1     2

1     1     1     3

1     1     3     4
       SEMIGROUP   NUMBER      22
       SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
             SUCH   THAT   YZ=Y


1     1     1     1

1     1     1     2

1     1     1     3

1     2     2     4
       SEMIGROUP   NUMBER      23
       SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
             SUCH   THAT   YZ=Y
```

```
1       1       1       1

1       1       1       2

1       1       1       3

1       2       3       4
         SEMIGROUP   NUMBER       24
         SEMIGROUP   IS  COMMUTATIVE
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 4
              SUCH   THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
              SUCH   THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       1       1       4

1       1       1       4

1       1       1       4

1       1       1       4
         SEMIGROUP   NUMBER       25


1       1       1       4

1       1       1       4

1       1       1       4

4       4       4       1
         SEMIGROUP   NUMBER       26
         SEMIGROUP   IS  COMMUTATIVE


1       1       1       4

1       1       1       4

1       1       1       4

4       4       4       4
         SEMIGROUP   NUMBER       27
         SEMIGROUP   IS  COMMUTATIVE


1       1       1       1

1       1       1       1

1       1       2       1

1       1       1       2
         SEMIGROUP   NUMBER       28
         SEMIGROUP   IS  COMMUTATIVE
```

```
1    1    1    1

1    1    1    1

1    1    2    1

1    1    1    4
     SEMIGROUP  NUMBER     29
     SEMIGROUP  IS  COMMUTATIVE


1    1    1    1

1    1    1    1

1    1    2    1

1    1    2    2
     SEMIGROUP  NUMBER     30


1    1    1    1

1    1    1    1

1    1    2    1

4    4    4    4
     SEMIGROUP  NUMBER     31


1    1    1    1

1    1    1    1

1    1    2    2

1    1    2    2
     SEMIGROUP  NUMBER     32
     SEMIGROUP  IS  COMMUTATIVE


1    1    1    1

1    1    1    2

1    1    2    3

1    2    3    4
     SEMIGROUP  NUMBER     33
     SEMIGROUP  IS  COMMUTATIVE
     SEMIGROUP  HAS  LEFT  IDENTITY  X = 4
           SUCH  THAT  XY=Y
     SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 4
           SUCH  THAT  YZ=Y
     SEMIGROUP  HAS  IDENTITY


1    1    1    4

1    1    1    4

1    1    2    4

1    1    1    4
     SEMIGROUP  NUMBER     34
```

```
1       1       1       4

1       1       1       4

1       1       2       4

4       4       4       1
         SEMIGROUP   NUMBER       35
         SEMIGROUP   IS   COMMUTATIVE

1       1       1       4

1       1       1       4

1       1       2       4

4       4       4       4
         SEMIGROUP   NUMBER       36
         SEMIGROUP   IS   COMMUTATIVE

1       1       1       1

1       1       1       1

1       1       3       1

1       1       1       4
         SEMIGROUP   NUMBER       37
         SEMIGROUP   IS   COMMUTATIVE

1       1       1       1

1       1       1       1

1       1       3       1

1       2       1       4
         SEMIGROUP   NUMBER       38

1       1       1       1

1       1       1       1

1       1       3       1

4       4       4       4
         SEMIGROUP   NUMBER       39

1       1       1       1

1       1       1       1

1       1       3       3

1       1       3       3
         SEMIGROUP   NUMBER       40
         SEMIGROUP   IS   COMMUTATIVE
```

1	1	1	1
1	1	1	1
1	1	3	3
1	1	3	4

SEMIGROUP NUMBER 41
SEMIGROUP IS COMMUTATIVE

1	1	1	1
1	1	1	1
1	1	3	3
1	1	4	4

SEMIGROUP NUMBER 42

1	1	1	1
1	1	1	1
1	1	3	3
1	2	3	4

SEMIGROUP NUMBER 43
SEMIGROUP HAS LEFT IDENTITY X = 4
SUCH THAT XY=Y

1	1	1	1
1	1	1	1
1	1	3	4
1	1	3	4

SEMIGROUP NUMBER 44

1	1	1	1
1	1	1	1
1	1	3	4
1	1	4	3

SEMIGROUP NUMBER 45
SEMIGROUP IS COMMUTATIVE

1	1	1	1
1	1	1	1
1	1	3	4
4	4	4	4

SEMIGROUP NUMBER 46

```
1       1       1       1

1       1       1       2

1       1       3       1

1       1       1       4
         SEMIGROUP  NUMBER      47


1       1       1       1

1       1       1       2

1       1       3       1

1       2       1       4
         SEMIGROUP  NUMBER      48
         SEMIGROUP  IS  COMMUTATIVE


1       1       1       1

1       1       1       2

1       1       3       3

1       1       3       4
         SEMIGROUP  NUMBER      49
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 4
              SUCH  THAT  YZ=Y


1       1       1       1

1       1       1       2

1       1       3       3

1       2       3       4
         SEMIGROUP  NUMBER      50
         SEMIGROUP  IS  COMMUTATIVE
         SEMIGROUP  HAS  LEFT  IDENTITY  X = 4
              SUCH  THAT  XY=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 4
              SUCH  THAT  YZ=Y
         SEMIGROUP  HAS  IDENTITY


1       1       1       4

1       1       1       4

1       1       3       4

1       1       1       4
         SEMIGROUP  NUMBER      51


1       1       1       4

1       1       1       4

1       1       3       4

1       1       4       4
         SEMIGROUP  NUMBER      52
```

```
1       1       1       4

1       1       1       4

1       1       3       4

4       4       4       1
         SEMIGROUP   NUMBER      53
         SEMIGROUP   IS  COMMUTATIVE

1       1       1       4

1       1       1       4

1       1       3       4

4       4       4       4
         SEMIGROUP   NUMBER      54
         SEMIGROUP   IS  COMMUTATIVE

1       1       1       1

1       1       1       1

1       2       3       1

4       4       4       4
         SEMIGROUP   NUMBER      55

1       1       1       1

1       1       1       1

1       2       3       3

1       2       3       3
         SEMIGROUP   NUMBER      56

1       1       1       1

1       1       1       1

1       2       3       3

1       2       3       4
         SEMIGROUP   NUMBER      57
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 4
              SUCH  THAT  XY=Y

1       1       1       1

1       1       1       1

1       2       3       3

1       2       4       4
         SEMIGROUP   NUMBER      58
```

```
1      1      1      1

1      1      1      1

1      2      3      4

1      2      3      4
         SEMIGROUP   NUMBER      59
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
               SUCH   THAT   XY=Y
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 4
               SUCH   THAT   XY=Y
```

```
1      1      1      1

1      1      1      1

1      2      3      4

1      2      4      3
         SEMIGROUP   NUMBER      60
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
               SUCH   THAT   XY=Y
```

```
1      1      1      1

1      1      1      1

1      2      3      4

4      4      4      4
         SEMIGROUP   NUMBER      61
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
               SUCH   THAT   XY=Y
```

```
1      1      1      1

1      1      1      2

1      2      3      1

1      1      1      4
         SEMIGROUP   NUMBER      62
```

```
1      1      1      1

1      1      1      2

1      2      3      2

1      1      1      4
         SEMIGROUP   NUMBER      63
```

1	1	1	1
1	1	1	2
1	2	3	3
1	2	3	4

SEMIGROUP NUMBER 64
SEMIGROUP HAS LEFT IDENTITY X = 4
SUCH THAT XY=Y
SEMIGROUP HAS RIGHT IDENTITY Z = 4
SUCH THAT YZ=Y
SEMIGROUP HAS IDENTITY

1	1	1	4
1	1	1	4
1	2	3	4
1	1	1	4

SEMIGROUP NUMBER 65
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y

1	1	1	4
1	1	1	4
1	2	3	4
1	1	4	4

SEMIGROUP NUMBER 66
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y

1	1	1	4
1	1	1	4
1	2	3	4
4	4	4	1

SEMIGROUP NUMBER 67
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y

1	1	1	4
1	1	1	4
1	2	3	4
4	4	4	4

SEMIGROUP NUMBER 68
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y

```
1     1     1     1
1     1     1     1
3     3     3     3
3     3     3     3
      SEMIGROUP   NUMBER       69

1     1     1     1
1     1     1     1
3     3     3     3
3     3     3     4
      SEMIGROUP   NUMBER       70

1     1     1     1
1     1     1     1
3     3     3     3
4     4     4     4
      SEMIGROUP   NUMBER       71

1     1     1     1
1     1     1     2
3     3     3     3
1     1     1     4
      SEMIGROUP   NUMBER       72
      SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
            SUCH   THAT   YZ=Y

1     1     1     1
1     1     1     2
3     3     3     3
1     1     3     4
      SEMIGROUP   NUMBER       73
      SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
            SUCH   THAT   YZ=Y

1     1     1     1
1     1     1     2
3     3     3     3
1     2     1     4
      SEMIGROUP   NUMBER       74
      SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
            SUCH   THAT   YZ=Y
```

```
1       1       1       1

1       1       1       2

3       3       3       3

1       2       3       4
         SEMIGROUP   NUMBER        75
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 4
                SUCH   THAT   XY=Y
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
                SUCH   THAT   YZ=Y
         SEMIGROUP   HAS   IDENTITY


1       1       1       1

1       1       1       2

3       3       3       3

3       3       3       4
         SEMIGROUP   NUMBER        76
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
                SUCH   THAT   YZ=Y


1       1       1       1

1       1       1       3

3       3       3       3

4       4       4       4
         SEMIGROUP   NUMBER        77


1       1       1       4

1       1       1       4

3       3       3       4

4       4       4       4
         SEMIGROUP   NUMBER        78


1       1       1       1

1       1       2       2

1       1       3       3

1       1       3       3
         SEMIGROUP   NUMBER        79


1       1       1       1

1       1       2       2

1       1       3       3

1       1       3       4
         SEMIGROUP   NUMBER        80
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 4
                SUCH   THAT   YZ=Y
```

```
1       1       1       1

1       1       2       2

1       1       3       3

1       1       4       4
         SEMIGROUP   NUMBER       81
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 3
              SUCH   THAT  YZ=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
              SUCH   THAT  YZ=Y


1       1       1       1

1       1       2       2

1       1       3       3

1       2       3       4
         SEMIGROUP   NUMBER       82
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 4
              SUCH   THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
              SUCH   THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       1       1       1

1       1       2       2

1       1       3       4

1       1       3       4
         SEMIGROUP   NUMBER       83


1       1       1       1

1       1       2       2

1       1       3       4

1       1       4       3
         SEMIGROUP   NUMBER       84
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 3
              SUCH   THAT  YZ=Y


1       1       1       4

1       1       2       4

1       1       3       4

1       1       1       4
         SEMIGROUP   NUMBER       85
```

```
1      1      1      4

1      1      2      4

1      1      3      4

1      1      4      4
        SEMIGROUP   NUMBER       86
        SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
             SUCH   THAT   YZ=Y
```

```
1      1      1      4

1      1      2      4

1      1      3      4

4      4      4      1
        SEMIGROUP   NUMBER       87
        SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
             SUCH   THAT   YZ=Y
```

```
1      1      1      4

1      1      2      4

1      1      3      4

4      4      4      4
        SEMIGROUP   NUMBER       88
        SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
             SUCH   THAT   YZ=Y
```

```
1      1      1      1

1      1      2      2

1      2      3      3

1      2      3      3
        SEMIGROUP   NUMBER       89
        SEMIGROUP   IS  COMMUTATIVE
```

```
1      1      1      1

1      1      2      2

1      2      3      3

1      2      3      4
        SEMIGROUP   NUMBER       90
        SEMIGROUP   IS  COMMUTATIVE
        SEMIGROUP   HAS  LEFT   IDENTITY   X = 4
             SUCH   THAT   XY=Y
        SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 4
             SUCH   THAT   YZ=Y
        SEMIGROUP   HAS  IDENTITY
```

1	1	1	1
1	1	2	2
1	2	3	3
1	2	4	4

SEMIGROUP NUMBER 91
SEMIGROUP HAS RIGHT IDENTITY Z = 3
SUCH THAT YZ=Y
SEMIGROUP HAS RIGHT IDENTITY Z = 4
SUCH THAT YZ=Y

1	1	1	1
1	1	2	2
1	2	3	4
1	2	3	4

SEMIGROUP NUMBER 92
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y
SEMIGROUP HAS LEFT IDENTITY X = 4
SUCH THAT XY=Y

1	1	1	1
1	1	2	2
1	2	3	4
1	2	4	3

SEMIGROUP NUMBER 93
SEMIGROUP IS COMMUTATIVE
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y
SEMIGROUP HAS RIGHT IDENTITY Z = 3
SUCH THAT YZ=Y
SEMIGROUP HAS IDENTITY

1	1	1	4
1	1	2	4
1	2	3	4
1	1	1	4

SEMIGROUP NUMBER 94
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y

1	1	1	4
1	1	2	4
1	2	3	4
1	1	4	4

SEMIGROUP NUMBER 95
SEMIGROUP HAS LEFT IDENTITY X = 3
SUCH THAT XY=Y
SEMIGROUP HAS RIGHT IDENTITY Z = 3
SUCH THAT YZ=Y
SEMIGROUP HAS IDENTITY

```
1       1       1       4

1       1       2       4

1       2       3       4

4       4       4       1
     SEMIGROUP   NUMBER      96
     SEMIGROUP   IS  COMMUTATIVE
     SEMIGROUP   HAS  LEFT  IDENTITY  X = 3
          SUCH   THAT  XY=Y
     SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 3
          SUCH   THAT  YZ=Y
     SEMIGROUP   HAS  IDENTITY


1       1       1       4

1       1       2       4

1       2       3       4

4       4       4       4
     SEMIGROUP   NUMBER      97
     SEMIGROUP   IS  COMMUTATIVE
     SEMIGROUP   HAS  LEFT  IDENTITY  X = 3
          SUCH   THAT  XY=Y
     SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 3
          SUCH   THAT  YZ=Y
     SEMIGROUP   HAS  IDENTITY


1       1       1       1

1       2       1       1

1       1       3       1

1       1       1       4
     SEMIGROUP   NUMBER      98
     SEMIGROUP   IS  COMMUTATIVE


1       1       1       1

1       2       1       1

1       1       3       1

4       4       4       4
     SEMIGROUP   NUMBER      99


1       1       1       1

1       2       1       1

1       1       3       3

1       1       3       3
     SEMIGROUP   NUMBER     100
     SEMIGROUP   IS  COMMUTATIVE
```

1	1	1	1
1	2	1	1
1	1	3	3
1	1	3	4

SEMIGROUP NUMBER 101
SEMIGROUP IS COMMUTATIVE

1	1	1	1
1	2	1	1
1	1	3	3
1	1	4	4

SEMIGROUP NUMBER 102

1	1	1	1
1	2	1	1
1	1	3	4
1	1	3	4

SEMIGROUP NUMBER 103

1	1	1	1
1	2	1	1
1	1	3	4
1	1	4	3

SEMIGROUP NUMBER 104
SEMIGROUP IS COMMUTATIVE

1	1	1	1
1	2	1	1
1	1	3	4
4	4	4	4

SEMIGROUP NUMBER 105

1	1	1	1
1	2	1	2
1	1	3	3
1	2	3	4

SEMIGROUP NUMBER 106
SEMIGROUP IS COMMUTATIVE
SEMIGROUP HAS LEFT IDENTITY X = 4
SUCH THAT XY=Y
SEMIGROUP HAS RIGHT IDENTITY Z = 4
SUCH THAT YZ=Y
SEMIGROUP HAS IDENTITY

1	1	1	4
1	2	1	4
1	1	3	4
1	1	1	4

SEMIGROUP NUMBER 107

1	1	1	4
1	2	1	4
1	1	3	4
1	1	4	4

SEMIGROUP NUMBER 108

1	1	1	4
1	2	1	4
1	1	3	4
4	4	4	1

SEMIGROUP NUMBER 109
SEMIGROUP IS COMMUTATIVE

1	1	1	4
1	2	1	4
1	1	3	4
4	4	4	4

SEMIGROUP NUMBER 110
SEMIGROUP IS COMMUTATIVE

1	1	1	1
1	2	1	1
3	3	3	3
3	3	3	4

SEMIGROUP NUMBER 111

1	1	1	1
1	2	1	1
3	3	3	3
4	4	4	4

SEMIGROUP NUMBER 112
SEMIGROUP HAS RIGHT IDENTITY Z = 2
SUCH THAT YZ=Y

```
1       1       1       1

1       2       1       2

3       3       3       3

1       2       1       2
         SEMIGROUP   NUMBER     113

1       1       1       1

1       2       1       2

3       3       3       3

1       2       1       4
         SEMIGROUP   NUMBER     114
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
              SUCH  THAT  YZ=Y

1       1       1       1

1       2       1       2

3       3       3       3

1       2       3       4
         SEMIGROUP   NUMBER     115
         SEMIGROUP   HAS  LEFT  IDENTITY  X = 4
              SUCH  THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
              SUCH  THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY

1       1       1       1

1       2       1       2

3       3       3       3

1       4       1       4
         SEMIGROUP   NUMBER     116
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
              SUCH  THAT  YZ=Y

1       1       1       1

1       2       1       2

3       3       3       3

3       4       3       4
         SEMIGROUP   NUMBER     117
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
              SUCH  THAT  YZ=Y
```

1	1	1	1
1	2	1	4
3	3	3	3
1	2	1	4

SEMIGROUP NUMBER 118

1	1	1	1
1	2	1	4
3	3	3	3
1	4	1	2

SEMIGROUP NUMBER 119
SEMIGROUP HAS RIGHT IDENTITY Z = 2
SUCH THAT YZ=Y

1	1	1	1
1	2	1	4
3	3	3	3
4	4	4	4

SEMIGROUP NUMBER 120
SEMIGROUP HAS RIGHT IDENTITY Z = 2
SUCH THAT YZ=Y

1	1	1	4
1	2	1	4
3	3	3	4
4	4	4	4

SEMIGROUP NUMBER 121
SEMIGROUP HAS RIGHT IDENTITY Z = 2
SUCH THAT YZ=Y

1	1	1	1
1	2	2	4
1	2	2	4
1	2	2	4

SEMIGROUP NUMBER 122

1	1	1	1
1	2	2	4
1	2	2	4
1	4	4	2

SEMIGROUP NUMBER 123
SEMIGROUP IS COMMUTATIVE

```
1     1     1     1
1     2     2     4
1     2     2     4
1     4     4     4
       SEMIGROUP  NUMBER    124
       SEMIGROUP  IS  COMMUTATIVE

1     1     1     1
1     2     2     4
1     2     2     4
4     4     4     4
       SEMIGROUP  NUMBER    125

1     1     1     4
1     2     2     4
1     2     2     4
1     1     1     4
       SEMIGROUP  NUMBER    126

1     1     1     4
1     2     2     4
1     2     2     4
1     4     4     4
       SEMIGROUP  NUMBER    127

1     1     1     4
1     2     2     4
1     2     2     4
4     4     4     1
       SEMIGROUP  NUMBER    128
       SEMIGROUP  IS  COMMUTATIVE

1     1     1     1
1     2     2     2
1     2     3     3
1     2     3     4
       SEMIGROUP  NUMBER    129
       SEMIGROUP  IS  COMMUTATIVE
       SEMIGROUP  HAS  LEFT  IDENTITY  X = 4
             SUCH  THAT  XY=Y
       SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 4
             SUCH  THAT  YZ=Y
       SEMIGROUP  HAS  IDENTITY
```

```
1       1       1       1

1       2       2       2

1       2       3       3

1       2       4       4
  SEMIGROUP   NUMBER     130
  SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
          SUCH  THAT   YZ=Y
  SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 4
          SUCH  THAT   YZ=Y


1       1       1       1

1       2       2       2

1       2       3       4

1       2       3       4
  SEMIGROUP   NUMBER     131
  SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
          SUCH  THAT   XY=Y
  SEMIGROUP   HAS  LEFT   IDENTITY   X = 4
          SUCH  THAT   XY=Y


1       1       1       1

1       2       2       2

1       2       3       4

1       2       4       3
  SEMIGROUP   NUMBER     132
  SEMIGROUP   IS  COMMUTATIVE
  SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
          SUCH  THAT   XY=Y
  SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
          SUCH  THAT   YZ=Y
  SEMIGROUP   HAS  IDENTITY


1       1       1       1

1       2       2       2

1       2       3       4

1       4       4       4
  SEMIGROUP   NUMBER     133
  SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
          SUCH  THAT   XY=Y
  SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
          SUCH  THAT   YZ=Y
  SEMIGROUP   HAS  IDENTITY


1       1       1       1

1       2       2       4

1       2       3       4

1       2       2       4
  SEMIGROUP   NUMBER     134
  SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
```

```
1       1       1       1

1       2       2       4

1       2       3       4

1       2       4       4
         SEMIGROUP   NUMBER      135
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 3
              SUCH    THAT   XY=Y
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 3
              SUCH    THAT   YZ=Y
         SEMIGROUP   HAS   IDENTITY

1       1       1       1

1       2       2       4

1       2       3       4

1       4       4       2
         SEMIGROUP   NUMBER      136
         SEMIGROUP   IS   COMMUTATIVE
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 3
              SUCH    THAT   XY=Y
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 3
              SUCH    THAT   YZ=Y
         SEMIGROUP   HAS   IDENTITY

1       1       1       1

1       2       2       4

1       2       3       4

4       4       4       4
         SEMIGROUP   NUMBER      137
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 3
              SUCH    THAT   XY=Y
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 3
              SUCH    THAT   YZ=Y
         SEMIGROUP   HAS   IDENTITY

1       1       1       4

1       2       2       4

1       2       3       4

1       1       1       4
         SEMIGROUP   NUMBER      138
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 3
              SUCH    THAT   XY=Y

1       1       1       4

1       2       2       4

1       2       3       4

1       1       4       4
         SEMIGROUP   NUMBER      139
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 3
              SUCH    THAT   XY=Y
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 3
              SUCH    THAT   YZ=Y
         SEMIGROUP   HAS   IDENTITY
```

```
1       1       1       4

1       2       2       4

1       2       3       4

1       4       4       4
         SEMIGROUP   NUMBER      140
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
              SUCH   THAT   XY=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
              SUCH   THAT   YZ=Y
         SEMIGROUP   HAS   IDENTITY


1       1       1       4

1       2       2       4

1       2       3       4

4       4       4       1
         SEMIGROUP   NUMBER      141
         SEMIGROUP   IS  COMMUTATIVE
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
              SUCH   THAT   XY=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
              SUCH   THAT   YZ=Y
         SEMIGROUP   HAS   IDENTITY


1       1       1       1

1       2       2       2

1       3       3       3

1       4       4       4
         SEMIGROUP   NUMBER      142
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 2
              SUCH   THAT   YZ=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
              SUCH   THAT   YZ=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 4
              SUCH   THAT   YZ=Y


1       1       1       1

1       2       2       4

1       3       3       4

4       4       4       4
         SEMIGROUP   NUMBER      143
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 2
              SUCH   THAT   YZ=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 3
              SUCH   THAT   YZ=Y

1       1       1       4

1       2       2       4

1       3       3       4

1       1       1       4
         SEMIGROUP   NUMBER      144
```

```
1       1       1       4

1       2       2       4

1       3       3       4

1       4       4       4
         SEMIGROUP   NUMBER      145
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 2
                 SUCH   THAT   YZ=Y
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 3
                 SUCH   THAT   YZ=Y


1       1       1       4

1       2       2       4

1       3       3       4

4       4       4       1
         SEMIGROUP   NUMBER      146
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 2
                 SUCH   THAT   YZ=Y
         SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 3
                 SUCH   THAT   YZ=Y


1       1       1       1

1       2       3       4

1       2       3       4

1       2       3       4
         SEMIGROUP   NUMBER      147
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
                 SUCH   THAT   XY=Y
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 3
                 SUCH   THAT   XY=Y
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 4
                 SUCH   THAT   XY=Y


1       1       1       1

1       2       3       4

1       2       3       4

4       4       4       4
         SEMIGROUP   NUMBER      148
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
                 SUCH   THAT   XY=Y
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 3
                 SUCH   THAT   XY=Y
```

```
1       1       1       4

1       2       3       4

1       2       3       4

1       1       1       4
          SEMIGROUP   NUMBER     149
          SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
                SUCH   THAT   XY=Y
          SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
                SUCH   THAT   XY=Y


1       1       1       4

1       2       3       4

1       2       3       4

1       4       4       4
          SEMIGROUP   NUMBER     150
          SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
                SUCH   THAT   XY=Y
          SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
                SUCH   THAT   XY=Y


1       1       1       4

1       2       3       4

1       2       3       4

4       4       4       1
          SEMIGROUP   NUMBER     151
          SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
                SUCH   THAT   XY=Y
          SEMIGROUP   HAS  LEFT   IDENTITY   X = 3
                SUCH   THAT   XY=Y


1       1       1       1

1       2       3       3

1       3       2       2

1       3       2       2
          SEMIGROUP   NUMBER     152
          SEMIGROUP   IS  COMMUTATIVE


1       1       1       1

1       2       3       4

1       3       2       4

4       4       4       4
          SEMIGROUP   NUMBER     153
          SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
                SUCH   THAT   XY=Y
          SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 2
                SUCH   THAT   YZ=Y
          SEMIGROUP   HAS  IDENTITY
```

```
1       1       1       4

1       2       3       4

1       3       2       4

1       1       1       4
    SEMIGROUP   NUMBER     154
    SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
          SUCH   THAT   XY=Y


1       1       1       4

1       2       3       4

1       3       2       4

1       4       4       4
    SEMIGROUP   NUMBER     155
    SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
          SUCH   THAT   XY=Y
    SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 2
          SUCH   THAT   YZ=Y
    SEMIGROUP   HAS   IDENTITY


1       1       1       4

1       2       3       4

1       3       2       4

4       4       4       1
    SEMIGROUP   NUMBER     156
    SEMIGROUP   IS   COMMUTATIVE
    SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
          SUCH   THAT   XY=Y
    SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 2
          SUCH   THAT   YZ=Y
    SEMIGROUP   HAS   IDENTITY


1       1       1       1

1       2       3       4

3       3       3       3

3       4       1       2
    SEMIGROUP   NUMBER     157
    SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
          SUCH   THAT   XY=Y
    SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 2
          SUCH   THAT   YZ=Y
    SEMIGROUP   HAS   IDENTITY
```

```
1       1       1       1

1       2       3       4

3       3       3       3

4       4       4       4
         SEMIGROUP  NUMBER     158
         SEMIGROUP  HAS  LEFT  IDENTITY  X = 2
              SUCH  THAT  XY=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y
         SEMIGROUP  HAS  IDENTITY


1       1       1       1

2       2       2       2

3       3       3       3

4       4       4       4
         SEMIGROUP  NUMBER     159
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 1
              SUCH  THAT  YZ=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 2
              SUCH  THAT  YZ=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 3
              SUCH  THAT  YZ=Y
         SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 4
              SUCH  THAT  YZ=Y


1       1       3       3

1       1       3       3

1       1       3       3

1       1       3       3
         SEMIGROUP  NUMBER     160


1       1       3       3

1       1       3       3

1       1       3       3

1       1       3       4
         SEMIGROUP  NUMBER     161


1       1       3       3

1       1       3       3

1       1       3       3

1       2       3       4
         SEMIGROUP  NUMBER     162
         SEMIGROUP  HAS  LEFT  IDENTITY  X = 4
              SUCH  THAT  XY=Y
```

```
1       1       3       4
1       1       3       4
1       1       3       4
1       1       3       4
         SEMIGROUP   NUMBER      163

1       1       3       4
1       1       3       4
1       1       3       4
1       3       3       4
         SEMIGROUP   NUMBER      164

1       1       3       3
1       1       3       3
3       3       1       1
3       3       1       1
         SEMIGROUP   NUMBER      165
         SEMIGROUP   IS   COMMUTATIVE

1       1       3       3
1       1       3       3
3       3       1       1
3       3       1       2
         SEMIGROUP   NUMBER      166
         SEMIGROUP   IS   COMMUTATIVE

1       1       3       3
1       2       3       3
1       1       3       3
1       1       3       4
         SEMIGROUP   NUMBER      167

1       1       3       3
1       2       3       4
1       1       3       3
1       2       3       4
         SEMIGROUP   NUMBER      168
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
                 SUCH   THAT   XY=Y
         SEMIGROUP   HAS   LEFT   IDENTITY   X = 4
                 SUCH   THAT   XY=Y
```

```
1       1       3       4

1       2       3       4

1       1       3       4

1       1       3       4
         SEMIGROUP   NUMBER     169
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
              SUCH    THAT   XY=Y


1       1       3       4

1       2       3       4

1       1       3       4

1       4       3       4
         SEMIGROUP   NUMBER     170
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
              SUCH    THAT   XY=Y


1       1       3       3

1       2       3       4

1       3       3       1

1       4       3       2
         SEMIGROUP   NUMBER     171
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
              SUCH    THAT   XY=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 2
              SUCH    THAT   YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       1       3       4

1       2       3       4

1       3       3       4

1       4       3       4
         SEMIGROUP   NUMBER     172
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
              SUCH    THAT   XY=Y
         SEMIGROUP   HAS  RIGHT   IDENTITY   Z = 2
              SUCH    THAT   YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       1       3       3

1       2       3       3

3       3       1       1

3       3       1       1
         SEMIGROUP   NUMBER     173
         SEMIGROUP   IS  COMMUTATIVE
```

```
1       1       3       3

1       2       3       3

3       3       1       1

3       4       1       1
         SEMIGROUP   NUMBER     174
         SEMIGROUP   HAS  RIGHT  IDENTITY   Z = 2
              SUCH   THAT  YZ=Y


1       1       3       3

1       2       3       4

3       3       1       1

3       3       1       1
         SEMIGROUP   NUMBER     175
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
              SUCH   THAT  XY=Y


1       1       3       3

1       2       3       4

3       3       1       1

3       4       1       1
         SEMIGROUP   NUMBER     176
         SEMIGROUP   IS  COMMUTATIVE
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
              SUCH   THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY   Z = 2
              SUCH   THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       1       3       3

1       2       3       4

3       3       1       1

3       4       1       2
         SEMIGROUP   NUMBER     177
         SEMIGROUP   IS  COMMUTATIVE
         SEMIGROUP   HAS  LEFT   IDENTITY   X = 2
              SUCH   THAT  XY=Y
         SEMIGROUP   HAS  RIGHT  IDENTITY   Z = 2
              SUCH   THAT  YZ=Y
         SEMIGROUP   HAS  IDENTITY


1       2       2       2

2       1       1       1

2       1       1       1

2       1       1       1
         SEMIGROUP   NUMBER     178
         SEMIGROUP   IS  COMMUTATIVE
```

```
1       2       3       4

1       2       3       4

1       2       3       4

1       2       3       4
  SEMIGROUP   NUMBER     179
  SEMIGROUP   HAS  LEFT  IDENTITY  X = 1
        SUCH  THAT  XY=Y
  SEMIGROUP   HAS  LEFT  IDENTITY  X = 2
        SUCH  THAT  XY=Y
  SEMIGROUP   HAS  LEFT  IDENTITY  X = 3
        SUCH  THAT  XY=Y
  SEMIGROUP   HAS  LEFT  IDENTITY  X = 4
        SUCH  THAT  XY=Y


1       2       3       1

2       3       1       2

3       1       2       3

1       2       3       1
  SEMIGROUP   NUMBER     180
  SEMIGROUP   IS  COMMUTATIVE


1       2       3       1

2       3       1       2

3       1       2       3

1       2       3       4
  SEMIGROUP   NUMBER     181
  SEMIGROUP   IS  COMMUTATIVE
  SEMIGROUP   HAS  LEFT  IDENTITY  X = 4
        SUCH  THAT  XY=Y
  SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 4
        SUCH  THAT  YZ=Y
  SEMIGROUP   HAS  IDENTITY


1       2       3       4

2       3       1       4

3       1       2       4

4       4       4       4
  SEMIGROUP   NUMBER     182
  SEMIGROUP   IS  COMMUTATIVE
  SEMIGROUP   HAS  LEFT  IDENTITY  X = 1
        SUCH  THAT  XY=Y
  SEMIGROUP   HAS  RIGHT  IDENTITY  Z = 1
        SUCH  THAT  YZ=Y
  SEMIGROUP   HAS  IDENTITY
```

```
1     2     3     4
1     2     3     4
3     4     1     2
3     4     1     2
  SEMIGROUP   NUMBER     183
  SEMIGROUP   HAS   LEFT   IDENTITY   X = 1
        SUCH   THAT   XY=Y
  SEMIGROUP   HAS   LEFT   IDENTITY   X = 2
        SUCH   THAT   XY=Y

1     2     2     4
2     4     4     1
2     4     4     1
4     1     1     2
  SEMIGROUP   NUMBER     184
  SEMIGROUP   IS   COMMUTATIVE

1     1     4     4
1     1     4     4
2     2     3     3
2     2     3     3
  SEMIGROUP   NUMBER     185

1     1     4     4
2     2     3     3
3     3     2     2
4     4     1     1
  SEMIGROUP   NUMBER     186
  SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 1
        SUCH   THAT   YZ=Y
  SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 2
        SUCH   THAT   YZ=Y

1     2     3     4
2     1     4     3
3     4     1     2
4     3     2     1
  SEMIGROUP   NUMBER     187
  SEMIGROUP   IS   COMMUTATIVE
  SEMIGROUP   HAS   LEFT   IDENTITY   X = 1
        SUCH   THAT   XY=Y
  SEMIGROUP   HAS   RIGHT   IDENTITY   Z = 1
        SUCH   THAT   YZ=Y
  SEMIGROUP   HAS   IDENTITY
  SEMIGROUP   IS   A   GROUP
```

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

```
SEMIGROUP  NUMBER    188
SEMIGROUP  IS  COMMUTATIVE
SEMIGROUP  HAS  LEFT  IDENTITY  X = 1
       SUCH  THAT  XY=Y
SEMIGROUP  HAS  RIGHT  IDENTITY  Z = 1
       SUCH  THAT  YZ=Y
SEMIGROUP  HAS  IDENTITY
SEMIGROUP  IS  A  GROUP
```

SEMIGROUPS OF ORDER FOUR WHICH ARE ISOMORPHIC TO THEIR ANTI-ISOMORPHIC IMAGE BUT NOT COMMUTATIVE

1	1	1	1
1	1	1	1
1	1	1	1
1	1	2	1

SEMIGROUP NUMBER 1

1	1	1	1
1	1	1	1
1	1	2	1
1	1	2	2

SEMIGROUP NUMBER 2

1	1	1	1
1	1	1	1
1	1	1	3
1	2	1	4

SEMIGROUP NUMBER 3

1	1	1	1
1	1	1	2
1	2	3	1
1	1	1	4

SEMIGROUP NUMBER 4

1	1	1	1
1	1	1	2
1	2	3	2
1	1	1	4

SEMIGROUP NUMBER 5

1	1	4	4
1	1	4	4
2	2	3	3
2	2	3	3

SEMIGROUP NUMBER 6

GENERATION OF ORDER TWO SEMIGROUPS

```
      INTEGER  H
      DIMENSION  I(10,10),M(20,3,3),L(5,5),LP(5,5),LF(5)
      N=2
      H=1
      M6=0
      DO  10  I1=1,N
      DO  10  J1=1,N
      DO  10  K1=1,N
      DO  10  L1=1,N
      I(1,1)=I1
      I(1,2)=J1
      I(2,1)=K1
      I(2,2)=L1
      L4=1
      N4=1
      I5=1
      L5=0
      N5=0
      DO  15  I2=1,N
      K2=N+1
      K2N=(N*N)+N
      DO  15  J2=K2,K2N
      IF  (J2.EQ.N+1)  L2=I1
      IF  (J2.EQ.N+2)  L2=J1
      IF  (J2.EQ.N+3)  L2=K1
      IF  (J2.EQ.N+4)  L2=L1
      I(I2,J2)=I(I2,L2)
   15 I(J2,I2)=I(L2,I2)
      DO  20  M3=1,N
      J3=(M3*N)+1
      J3N=(M3*N)+N
      DO  20  K3=J3,J3N
      DO  20  L3=1,N
      I3=(((K3-(M3*N))*(N-L3))+((K3-(M3*N)+1)*L3))
      IF  (I(M3,I3).NE.I(K3,L3))  GO  TO  24
   20 CONTINUE
      WRITE  (6,21)  H,((I(J,K),K=1,2),J=1,2)
      WRITE  (7,210)   ((I(J,K),K=1,2),J=1,2)
   21 FORMAT  (///1X'PERMUTATION  NUMBER',I5,'  IS  A  SEMI
     1GROUP',//2X,I2,4X,I2,//2X,I2,4X,I2)
  210 FORMAT  (///44X,//2X,I2,4X,I2,//2X,I2,4X,I2)
      M6=M6+1
      DO  16  I6=1,N
      DO  16  J6=1,N
   16 M(M6,I6,J6)=I(I6,J6)
      WRITE  (6,17)  M6
      WRITE  (7,17)  M6
   17 FORMAT  (10X'SEMIGROUP  NUMBER  IS ',I5)
      IF  (M6.EQ.1)  GO  TO  19
      LF(1)=2
      LF(2)=1
      DO  18  N6=1,N
      DO  18  L6=1,N
      DO  18  K6=1,N
      IF  (M(M6,L6,K6).EQ.N6)  L(L6,K6)=LF(N6)
   18 CONTINUE
      LP(1,1)=L(2,2)
      LP(1,2)=L(2,1)
      LP(2,1)=L(1,2)
      LP(2,2)=L(1,1)
      L7=1
   11 DO  12  I7=1,N
      DO  12  J7=1,N
      IF  (LP(I7,J7).NE.M(L7,I7,J7))  GO  TO  13
```

```
12 CONTINUE
   GO  TO  14
13 L7=L7+1
   IF  (L7.EQ.M6)  GO  TO  19
   GO  TO  11
14 WRITE  (6,8)  L7
 8 FORMAT  (10X'SEMIGROUP  IS  ISOMORPHIC  TO  SEMIGROUP
  1  NUMBER',I5)
   GO  TO  24
19 DO  22  I4=1,N
   DO  22  J4=1,N
   IF  (I(I4,J4).NE.I(J4,I4))  GO  TO  26
22 CONTINUE
   WRITE  (6,23)
   WRITE  (7,23)
23 FORMAT  (10X'SEMIGROUP  IS  COMMUTATIVE')
26 DO  27  K4=1,N
   IF  (I(L4,K4).NE.K4)  GO  TO  29
27 CONTINUE
   WRITE  (6,28)  L4
   WRITE  (7,28)  L4
   L5=L4
28 FORMAT  (10X'SEMIGROUP  HAS  LEFT  IDENTITY  X =',I2,
  1'  SUCH  THAT  XY=Y')
29 L4=L4+1
   IF  (L4.GT.N)  GO  TO  30
   GO  TO  26
30 DO  31  M4=1,N
   IF  (I(M4,N4).NE.M4)  GO  TO  33
31 CONTINUE
   WRITE  (6,32)  N4
   WRITE  (7,32)  N4
   N5=N4
32 FORMAT  (10X'SEMIGROUP  HAS  RIGHT  IDENTITY  Z =',I2
  1,'  SUCH  THAT  YZ=Y')
33 N4=N4+1
   IF  (N4.GT.N)  GO  TO  34
   GO  TO  30
34 IF  ((N5.NE.L5).OR.((N5.EQ.0).AND.(L5.EQ.0)))
  1GO  TO  24
   WRITE  (6,35)
   WRITE  (7,35)
35 FORMAT  (10X'SEMIGROUP  HAS  IDENTITY')
36 DO  37  J5=1,N
   IF  (I(I5,J5).EQ.N5)  GO  TO  38
37 CONTINUE
   GO  TO  24
38 I5=I5+1
   IF  (I5.GT.N)  GO  TO  39
   GO  TO  36
39 WRITE  (6,40)
   WRITE  (7,40)
40 FORMAT  (10X'SEMIGROUP  IS  A  GROUP')
24 H=H+1
10 CONTINUE
68 STOP
   END
```

GENERATION OF ORDER THREE SEMIGROUPS

```
      INTEGER  H
      DIMENSION  I(15,15),M(400,5,5),LP(5,5),L(5,5),LF(5)
     1 ,LR(5)
      N=3
      H=1
      M8=0
      M9=0
      DO  10  I1=1,N
      DO  10  I2=1,N
      DO  10  I3=1,N
      DO  10  I4=1,N
      DO  10  I5=1,N
      DO  10  I6=1,N
      DO  10  I7=1,N
      DO  10  I8=1,N
      DO  10  I9=1,N
      L4=1
      N4=1
      L5=0
      N5=0
      J6=1
      I(1,1)=I1
      I(1,2)=I2
      I(1,3)=I3
      I(2,1)=I4
      I(2,2)=I5
      I(2,3)=I6
      I(3,1)=I7
      I(3,2)=I8
      I(3,3)=I9
      DO  15  J1=1,N
      J2=N+1
      J2N=(N*N)+N
      DO  15  J3=J2,J2N
      IF  (J3.EQ.N+1)  J4=I1
      IF  (J3.EQ.N+2)  J4=I2
      IF  (J3.EQ.N+3)  J4=I3
      IF  (J3.EQ.N+4)  J4=I4
      IF  (J3.EQ.N+5)  J4=I5
      IF  (J3.EQ.N+6)  J4=I6
      IF  (J3.EQ.N+7)  J4=I7
      IF  (J3.EQ.N+8)  J4=I8
      IF  (J3.EQ.N+9)  J4=I9
      I(J1,J3)=I(J1,J4)
   15 I(J3,J1)=I(J4,J1)
      DO  20  M1=1,N
      M2=(M1*N)+1
      M2N=(M1*N)+N
      DO  20  M3=M2,M2N
      DO  20  M4=1,N
      M5=(((M3-(M1*N))*(N-M4))+((M3-(M1*N)+1)*M4))
      IF  (I(M1,M5).NE.I(M3,M4))  GO  TO  24
   20 CONTINUE
      WRITE  (6,21)  H,((I(J,K),K=1,3),J=1,3)
   21 FORMAT  (///1X'PERMUTATION  NUMBER',I5,'  IS  A  SEMI
     1  GROUP',//2X,I2,4X,I2,4X,I2,//2X,I2,4X,I2,4X,I2,//2X,
     2I2,4X,I2,4X,I2)
      M9=M9+1
      M8=M8+1
      DO  50  I10=1,N
      DO  50  J10=1,N
   50 M(M9,I10,J10)=I(I10,J10)
      WRITE  (6,51)  M9
      WRITE  (7,51)  M8
```

```
 51 FORMAT  (10X'SEMIGROUP  NUMBER',I5)
    IF  ((M9.EQ.1).OR.(M9.EQ.400))  GO  TO  5
    DO  90  I20=1,N
    DO  90  I21=1,N
    DO  90  I22=1,N
    LF(1)=I20
    LF(2)=I21
    LF(3)=I22
    LR(I20)=1
    LR(I21)=2
    LR(I22)=3
    IF  ((I20.EQ.I21).OR.(I20.EQ.I22).OR.(I21.EQ.I22))
   1 GO  TO  90
 74 DO  80  K10=1,N
    DO  80  M10=1,N
    DO  80  N10=1,N
    IF  (I(M10,N10).EQ.K10)  L(M10,N10)=LF(K10)
 80 CONTINUE
    LP(1,1)=L(LR(1),LR(1))
    LP(1,2)=L(LR(1),LR(2))
    LP(1,3)=L(LR(1),LR(3))
    LP(2,1)=L(LR(2),LR(1))
    LP(2,2)=L(LR(2),LR(2))
    LP(2,3)=L(LR(2),LR(3))
    LP(3,1)=L(LR(3),LR(1))
    LP(3,2)=L(LR(3),LR(2))
    LP(3,3)=L(LR(3),LR(3))
 79 L20=1
 59 DO  60  I11=1,N
    DO  60  J11=1,N
    IF  (LP(I11,J11).NE.M(L20,I11,J11))  GO  TO  61
 60 CONTINUE
    GO  TO  65
 61 L20=L20+1
    IF  (L20.EQ.M9)  GO  TO  90
    GO  TO  59
 65 WRITE  (6,66)  L20
 66 FORMAT  (10X'SEMIGROUP  IS  ISOMORPHIC  TO  SEMIGROUP
   1  NUMBER',I5)
    M8=M8-1
    GO  TO  10
 90 CONTINUE
    DO  500  I500=1,N
    J500=N+1-I500
    DO  501  I501=1,N
    DO  501  I502=1,N
    IF  ((I501.EQ.J500).OR.(I502.EQ.J500))  GO  TO  501
    IF  (I(I501,I502).EQ.J500)  GO  TO  500
501 CONTINUE
    N501=1
    N502=2
    N503=3
    IF  (J500.EQ.1)  GO  TO  503
    IF  (J500.EQ.2)  GO  TO  504
    IF  (J500.EQ.3)  GO  TO  505
503 WRITE  (7,502)  N502,N503
    GO  TO  500
504 WRITE  (7,502)  N501,N503
    GO  TO  500
505 WRITE  (7,502)  N501,N502
502 FORMAT  (10X'SEMIGROUP  HAS  A  SUBSEMIGROUP  OF
   1  ORDER  TWO',//14X'(',I1,',',I1,')')
500 CONTINUE
  5 DO  22  K1=1,N
    DO  22  K2=1,N
    IF  (I(K1,K2).NE.I(K2,K1))  GO  TO  26
 22 CONTINUE
    WRITE  (6,23)
 23 FORMAT  (10X'SEMIGROUP  IS  COMMUTATIVE')
 26 DO  27  K4=1,N
    IF  (I(L4,K4).NE.K4)  GO  TO  29
 27 CONTINUE
```

```
      WRITE  (6,28)  L4
      WRITE  (7,28)  L4
      L5=L4
   28 FORMAT  (10X'SEMIGROUP  HAS  LEFT  IDENTITY  X =',I2,
     1//14X'  SUCH  THAT  XY=Y')
   29 L4=L4+1
      IF  (L4.GT.N)  GO  TO  30
      GO  TO  26
   30 DO  31  M6=1,N
      IF  (I(M6,N4).NE.M6)  GO  TO  33
   31 CONTINUE
      WRITE  (6,32)  N4
      WRITE  (7,32)  N4
      N5=N4
   32 FORMAT  (10X'SEMIGROUP  HAS  RIGHT  IDENTITY  Z =',I2,
     1//14X'  SUCH  THAT  YZ=Y')
   33 N4=N4+1
      IF  (N4.GT.N)  GO  TO  34
      GO  TO  30
   34 IF  ((N5.NE.L5).OR.((N5.EQ.0).AND.(L5.EQ.0)))
     1  GO  TO  24
      WRITE  (6,35)
   35 FORMAT  (10X'SEMIGROUP  HAS  IDENTITY')
   36 DO  37  J5=1,N
      IF  (I(J6,J5).EQ.N5)  GO  TO  38
   37 CONTINUE
      GO  TO  24
   38 J6=J6+1
      IF  (J6.GT.N)  GO  TO  39
      GO  TO  36
   39 WRITE  (6,40)
   40 FORMAT  (10X'SEMIGROUP  IS  A  GROUP')
   24 H=H+1
      IF  (H.GT.20000)  GO  TO  68
   10 CONTINUE
   68 WRITE  (6,25)  H
   25 FORMAT  (///5X'H  =',I5)
      STOP
      END
```

```
      INTEGER  H
      DIMENSION  I(10,10),J(15,10,10)
      H=1
      N=4
      N1=1
      N2=1
      I13=9
      I14=9
      I24=9
      I31=9
      I33=9
      I34=9
      I41=9
      I42=9
      I43=9
      I44=9
      DO   600   K3=1,9
      DO   600   K4=1,9
      I(K3,K4)=9
  600 CONTINUE
      DO   100   I11=1,N1
      DO   100   I12=1,N
      DO   100   I21=1,N
      DO   100   I22=1,N
      DO   100   I23=1,N
      DO   100   I32=1,N
      I(1,1)=I11
      I(1,2)=I12
      I(1,3)=I13
      I(1,4)=I14
      I(2,1)=I21
      I(2,2)=I22
      I(2,3)=I23
      I(2,4)=I24
      I(3,1)=I31
      I(3,2)=I32
      I(3,3)=I33
      I(3,4)=I34
      I(4,1)=I41
      I(4,2)=I42
      I(4,3)=I43
      I(4,4)=I44
      DO   800   L1=1,3
      DO   800   L2=1,3
      DO   800   L3=1,3
      IF   ((I(L1,L2).EQ.9).OR.(I(L2,L3).EQ.9))   GO   TO   800
      IF   (I(L1,I(L2,L3)).NE.I(I(L1,L2),L3))   GO   TO   801
      GO   TO   800
  801 IF   ((I(L1,I(L2,L3)).EQ.9).OR.(I(I(L1,L2),L3).EQ.9))
     1  GO   TO   800
      GO   TO   100
  800 CONTINUE
      L7=10*H
      WRITE   (6,14)   L7,((I(K,L),L=1,4),K=1,4)
      N7=0
      N9=0
      N12=0
      N13=0
  300 DO   500   KK=1,14
      DO   500   K1=1,N
      DO   500   K2=1,N
      J(KK,K1,K2)=I(K1,K2)
  500 CONTINUE
      IF   (N7.EQ.0)   GO   TO   999
```

```
      I(3,3)=N7
      J(1,3,3)=N7
      IF  (N12.GE.1)  GC  TC  852
  808 DC  850  LL=1,N
      DO  850  MM=1,N
      IF  (J(1,LL,MM).GE.5)  GO  TO  851
      GC  TO  850
  851 N13=N13+1
      IF  (N13.GE.7)  GC  TO  852
  850 CONTINUE
      GC  TO  902
  852 N12=N12+1
      IF  (N12.GE.5)  GC  TC  853
      J(1,LL,MM)=N12
      I(LL,MM)=N12
      GO  TO  902
  853 N13=0
      N12=0
      GC  TO  905
  902 J(13,I(1,3),3)=I(1,I(3,3))
      J(13,I23,3)=I(2,I(3,3))
      J(13,I(3,3),3)=I(3,I(3,3))
      J(13,I(4,3),3)=I(4,I(3,3))
      J(14,3,I(3,1))=I(I(3,3),1)
      J(14,3,I32)=I(I(3,3),2)
      J(14,3,I(3,4))=I(I(3,3),4)
      J(14,3,I(3,3))=I(I(3,3),3)
      IF  (N7.NE.4)  GO  TO  999
      J(12,4,1)=I(3,I(3,1))
      J(12,4,2)=I(3,I(3,2))
      J(12,4,3)=I(3,I(3,3))
      J(12,4,4)=I(3,I(3,4))
      J(12,3,4)=I(I(3,3),3)
      J(12,2,4)=I(I(2,3),3)
      J(12,1,4)=I(I(1,3),3)
      IF  (J(12,4,4).EQ.9)  J(12,4,4)=I(I(4,3),3)
      GO  TO  999
  999 IF  (I12-3)19,20,21
   19 GO  TO  22
   20 J(1,I12,1)=I(1,I21)
      J(1,I12,3)=I(1,I23)
      J(1,I12,4)=I(1,I(2,4))
      J(1,1,I12)=I(I11,2)
      J(2,3,I12)=I(I(3,1),2)
      J(1,4,I12)=I(I(4,1),2)
   21 J(1,I12,1)=I(1,I21)
      J(1,I12,2)=I(1,I22)
      J(2,I12,3)=I(1,I23)
      J(1,I12,4)=I(1,I(2,4))
      J(1,1,I12)=I(I11,2)
      J(1,2,I12)=I(I21,2)
      J(2,3,I12)=I(I(3,1),2)
      J(2,4,I12)=I(I(4,1),2)
      N2=2
      GC  TO  30
   22 IF  (I22-3)23,24,25
   23 GO  TO  26
   24 J(2,I22,1)=I(2,I21)
      J(3,I22,3)=I(2,I23)
      J(3,I22,4)=I(2,I(2,4))
      J(2,1,I22)=I(I12,2)
      J(4,3,I22)=I(I32,2)
      J(3,4,I22)=I(I(4,2),2)
   25 J(2,I22,1)=I(2,I21)
      J(2,I22,2)=I(2,I22)
      J(4,I22,3)=I(2,I23)
      J(3,I22,4)=I(2,I(2,4))
      J(2,1,I22)=I(I12,2)
      J(2,2,I22)=I(I22,2)
      J(4,3,I22)=I(I32,2)
      J(4,4,I22)=I(I(4,2),2)
      N2=4
```

```
      GO  TO  30
   26 IF(I21-3)27,28,29
   27 GC  TO  31
   28 J(3,I21,1)=I(2,I11)
      J(5,I21,3)=I(2,I(1,3))
      J(5,I21,4)=I(2,I(1,4))
      J(3,1,I21)=I(I12,1)
      J(6,3,I21)=I(I32,1)
      J(5,4,I21)=I(I(4,2),1)
   29 J(3,I21,1)=I(2,I11)
      J(3,I21,2)=I(2,I12)
      J(6,I21,3)=I(2,I(1,3))
      J(5,I21,4)=I(2,I(1,4))
      J(3,1,I21)=I(I12,1)
      J(3,2,I21)=I(I22,1)
      J(6,3,I21)=I(I32,1)
      J(6,4,I21)=I(I(4,2),1)
      N2=6
      GC  TO  30
   31 IF(I23-3)32,33,34
   32 GO  TO  35
   33 J(4,I23,1)=I(2,I(3,1))
      J(7,I23,3)=I(2,I(3,3))
      J(7,I23,4)=I(2,I(3,4))
      J(4,1,I23)=I(I12,3)
      J(8,3,I23)=I(I32,3)
      J(7,4,I23)=I(I(4,2),3)
   34 J(4,I23,1)=I(2,I(3,1))
      J(4,I23,2)=I(2,I32)
      J(8,I23,3)=I(2,I(3,3))
      J(7,I23,4)=I(2,I(3,4))
      J(4,1,I23)=I(I12,3)
      J(4,2,I23)=I(I22,3)
      J(8,3,I23)=I(I32,3)
      J(8,4,I23)=I(I(4,2),3)
      N2=8
   30 N3=N2/2
      DO  300  L3=N3,N2
      DO  300  L4=1,N
      DO  300  L5=1,N
      IF  (J(L3,L4,L5).NE.9)  I(L4,L5)=J(L3,L4,L5)
  300 CONTINUE
      IF  (N2.EQ.8)  GO TO  35
      IF  (N2-4)22,26,31
   35 IF  (I32-3)36,37,38
   36 GO  TO  39
   37 J(5,I32,1)=I(3,I21)
      J(9,I32,3)=I(3,I23)
      J(9,I32,4)=I(3,I(2,4))
      J(5,1,I32)=I(I(1,3),2)
      J(10,3,I32)=I(I(3,3),2)
      J(9,4,I32)=I(I(4,3),2)
   38 J(5,I32,1)=I(3,I21)
      J(5,I32,2)=I(3,I22)
      J(10,I32,3)=I(3,I23)
      J(9,I32,4)=I(3,I(2,4))
      J(5,1,I32)=I(I(1,3),2)
      J(5,2,I32)=I(I23,2)
      J(10,3,I32)=I(I(3,3),2)
      J(10,4,I32)=I(I(4,3),2)
      N2=10
   39 N3=N2/2
      IF  (N3.EQ.0)  GO  TO  18
      DO  301  L3=N3,N2
      DO  301  L4=1,N
      DO  301  L5=1,N
      IF  (J(L3,L4,L5).NE.9)  I(L4,L5)=J(L3,L4,L5)
  301 CONTINUE
   18 IF  (I12.EQ.1)  GC  TC  40
      IF  (I12.EQ.2)  GC  TC  41
   40 J(6,I12,3)=I(1,I23)
      J(6,I12,4)=I(1,I(2,4))
```

```
      J(6,3,I12)=I(I(3,1),2)
      J(6,4,I12)=I(I(4,1),2)
   41 J(6,I12,4)=I(1,I(2,4))
      J(6,4,I12)=I(I(4,1),2)
      N6=6
      GO  TO  50
   51 IF  (I21.EQ.1)  GO  TO  42
      IF  (I21.EQ.2)  GO  TO  43
   42 J(7,I21,3)=I(2,I(1,3))
      J(7,I21,4)=I(2,I(1,4))
      J(7,3,I21)=I(I32,1)
      J(7,4,I21)=I(I(4,2),1)
   43 J(7,I21,4)=I(2,I(1,4))
      J(7,4,I21)=I(I(4,2),1)
      N6=7
      GO  TO  50
   52 IF  (I22.EQ.1)  GO  TO  44
      IF  (I22.EQ.2)  GO  TO  45
   44 J(8,I22,3)=I(2,I23)
      J(8,I22,4)=I(2,I(2,4))
      J(8,3,I22)=I(I32,2)
      J(8,4,I22)=I(I(4,2),2)
   45 J(8,I22,4)=I(2,I(2,4))
      J(8,4,I22)=I(I(4,2),2)
      N6=8
      GO  TO  50
   54 IF  (I23.EQ.1)  GO  TO  46
      IF  (I23.EQ.2)  GO  TO  47
   46 J(9,I23,3)=I(2,I(3,3))
      J(9,I23,4)=I(2,I(3,4))
      J(9,3,I23)=I(I32,3)
      J(9,4,I23)=I(I(4,2),3)
   47 J(9,I23,4)=I(2,I(3,4))
      J(9,4,I23)=I(I(4,2),3)
      N6=9
      GO  TO  50
   55 IF  (I32.EQ.1)  GO  TO  48
      IF  (I32.EQ.2)  GO  TO  49
   48 J(10,I32,3)=I(3,I23)
      J(10,I32,4)=I(3,I(2,4))
      J(10,3,I32)=I(I(3,3),2)
      J(10,4,I32)=I(I(4,3),2)
   49 J(10,I32,4)=I(3,I(2,4))
      J(10,4,I32)=I(I(4,3),2)
      N6=10
   50 DO  302  L4=1,N
      DO  302  L5=1,N
      IF  (J(N6,L4,L5).NE.9)  I(L4,L5)=J(N6,L4,L5)
  302 CONTINUE
      IF  (N6-7)51,52,53
   53 IF  (N6-9)54,55,56
   56 DO  400  L4=1,N
      DO  400  L5=1,N
      IF  (I(L4,L5).EQ.4)  GO  TO  401
      IF  (I(L4,L5).EQ.3)  GO  TO  402
      GO  TO  400
  401 J(11,4,1)=I(L4,I(L5,1))
      J(11,4,2)=I(L4,I(L5,2))
      J(11,4,3)=I(L4,I(L5,3))
      J(11,4,4)=I(L4,I(L5,4))
      J(11,3,4)=I(I(3,L4),L5)
      J(11,2,4)=I(I(2,L4),L5)
      J(11,1,4)=I(I(1,L4),L5)
      IF(J(11,4,4).EQ.9)  J(11,4,4)=I(I(4,L4),L5)
      GO  TO  400
  402 J(15,3,1)=I(L4,I(L5,1))
      J(15,3,2)=I(L4,I(L5,2))
      J(15,3,4)=I(L4,I(L5,4))
      J(15,1,3)=I(I(1,L4),L5)
      J(15,2,3)=I(I(2,L4),L5)
      J(15,4,3)=I(I(4,L4),L5)
      J(15,3,3)=I(I(3,L4),L5)
```

```
      IF  (J(15,3,3).EQ.9)  J(15,3,3)=I(L4,I(L5,3))
  400 CONTINUE
      DO  8  L4=1,N
      DO  8  L5=1,N
      N4=2
      IF  (J(1,L4,L5)-9)8,10,10
   10 J(1,L4,L5)=J(N4,L4,L5)
      IF  (J(N4,L4,L5)-9)8,11,11
   11 N4=N4+1
      IF  (N4.EQ.13)  GO  TO  14
      IF  (N4.EQ.14)  GO  TO  8
      GO TO 10
   14 IF  (J(13,L4,L5).EQ.9)  GO  TO  13
      GO  TO  10
   13 J(1,L4,L5)=I(L4,L5)
    8 CONTINUE
      WRITE  (6,105)  N7
  105 FORMAT  (20X,I5)
      L8=100*H
      WRITE  (6,16) L8,((J(1,K,L),L=1,4),K=1,4)
      GO  TO  909
  905 IF  (N7-1)904,903,903
  904 IF  (J(1,3,3).GE.5)  GO  TO  903
      GO  TO  909
  903 IF  (N12.NE.0)  GO  TO  900
      N7=N7+1
      IF  (N7.GE.5)  GO  TO  102
      GO  TO  900
  909 N5=1
      DO  9  L4=1,N
      DO  9  L5=1,N
      IF  (J(1,L4,L5).NE.9)   GO  TO  15
      IF  (N5-2)5,6,7
    5 J(1,L4,L5)=5
      N5=N5+1
      GO  TO  9
    6 J(1,L4,L5)=6
      N5=N5+1
      GO  TO  9
    7 IF  (N5-4)12,4,3
   12 J(1,L4,L5)=7
      N5=N5+1
      GO  TO  9
    4 J(1,L4,L5)=15
      N5=N5+1
      GO  TO  9
    3 IF  (N5-6)2,1,9
    2 J(1,L4,L5)=16
      N5=N5+1
      GO  TO  9
    1 J(1,L4,L5)=17
      GO  TO  9
   15 CONTINUE
    9 CONTINUE
      DO  200  L1=1,N
      DO  200  L2=1,N
      IF  (J(1,L1,L2).EQ.9)  GO  TO  201
      GO  TO  200
  201 J(1,L1,L2)=I(L1,L2)
  200 CONTINUE
      IF  (N9.EQ.0)  GO  TO  710
      WRITE  (7,777)   ((J(1,K,L),L=1,4),K=1,4)
  777 FORMAT  (I15,16I2)
      WRITE  (6,16)  H,((J(1,K,L),L=1,4),K=1,4)
   16 FORMAT  (10X'PERMUTATION',I5,//2X,I2,4X,I2,4X,I2,4X,I2
     1,//2X,I2,4X,I2,4X,I2,4X,I2,//2X,I2,4X,I2,4X,I2,4X,I2,/
     2/2X,I2,4X,I2,4X,I2,4X,I2,//2X,I2,4X,I2,4X,I2,4X,I2)
  710 CONTINUE
      I(1,1)=I11
      I(1,2)=I12
      I(1,3)=I13
      I(1,4)=I14
```

```
      I(2,1)=I21
      I(2,2)=I22
      I(2,3)=I23
      I(2,4)=I24
      I(3,1)=I31
      I(3,2)=I32
      I(3,3)=I33
      I(3,4)=I34
      I(4,1)=I41
      I(4,2)=I42
      I(4,3)=I43
      I(4,4)=I44
      IF  (N7.EQ.0)  GO  TO  707
      GO  TO  905
  707 N9=N9+1
      IF  (N9.GE.2)  GO  TO  102
      GO  TO  905
  102 CONTINUE
      H=H+1
      IF  (H.EQ.50)  GO  TO  101
  100 CONTINUE
  101 CONTINUE
      STOP
      END
```

GENERATION OF ORDER FOUR SEMIGROUPS PART TWO

```
      INTEGER  H
      DIMENSION  I(70,70),J(70,70),J1(5,5,5)
      N=4
      H=1
      DO  10  K9=1,200
      READ  (5,11)  M,((I(L,K),K=1,4),L=1,4)
   11 FORMAT  (17I4)
      IF  (M.EQ.9999)  GO  TO  10
      N12=0
      DO  60  LL=1,N
      DO  60  KK=1,N
      IF  (I(LL,KK).GE.5)  GO  TO  61
      GO  TO  60
   61 N12=N12+1
      IF  (N12.GE.6)  GO  TO  107
   60 CONTINUE
      DO  100  I21=1,N
      DO  100  I22=1,N
      DO  100  I23=1,N
      DO  101  I24=1,N
      DO  101  I25=1,N
  101 J(I24,I25)=I(I24,I25)
      DO  50  I11=1,N
      DO  50  I12=1,N
  102 IF  (I(I11,I12).EQ.5)  GO  TO  51
      IF  (I(I11,I12).EQ.6)  GO  TO  52
      IF  (I(I11,I12).EQ.7)  GO  TO  53
      GO  TO  49
   51 J(I11,I12)=I21
      GO  TO  49
   52 J(I11,I12)=I22
      GO  TO  49
   53 J(I11,I12)=I23
   49 IF  (J(I11,I12).LE.4)  GO  TO  50
      I(I11,I12)=J(I11,I12)
      GO  TO  102
   50 CONTINUE
  333 DO  105  M3=1,N
      DO  105  M4=1,N
      J1(1,M3,M4)=J(M3,M4)
  105 CONTINUE
      DO  12  K1=1,N
      DO  12  K2=1,N
      DO  12  K3=1,N
      IF  (J(J(K1,K2),K3).NE.J(K1,J(K2,K3)))  GO  TO  24
   12 CONTINUE
      IF  (H.EQ.1)  GO  TO  103
      DO  104  M1=1,N
      DO  104  M2=1,N
      IF  (J1(2,M1,M2).NE.J1(1,M1,M2))  GO  TO  103
  104 CONTINUE
      GO  TO  24
  103 WRITE  (6,21)  H,((J(L,K),K=1,4),L=1,4)
      WRITE  (7,21)  H,((J(L,K),K=1,4),L=1,4)
      WRITE  (6,22)  M
   21 FORMAT  (I15,16I2)
   22 FORMAT  (I60)
   24 H=H+1
      DO  106  M5=1,N
      DO  106  M6=1,N
      J1(2,M5,M6)=J1(1,M5,M6)
  106 CONTINUE
  100 CONTINUE
      GO  TO  10
```

```
107 I(1,1)=1
    DO   120   I21=1,N
    DO   120   I22=1,N
    DO   120   I23=1,N
    DO   120   I24=1,N
    DO   120   I25=1,N
    DO   120   I26=1,N
    DO   109   L8=1,N
    DO   109   L9=1,N
109 J(L8,L9)=I(L8,L9)
    DO   500  I11=1,N
    DO   500  I12=1,N
111 IF   (I(I11,I12).EQ.5)   GO   TO   510
    IF   (I(I11,I12).EQ.6)   GO   TO   520
    IF   (I(I11,I12).EQ.7)   GO   TO   530
    IF   (I(I11,I12).EQ.15)   GO   TO   690
    IF   (I(I11,I12).EQ.16)   GO   TO   700
    IF   (I(I11,I12).EQ.17)   GO   TO   710
    GO   TO   490
510 J(I11,I12)=I21
    GO   TO   490
520 J(I11,I12)=I22
    GO   TO   490
530 J(I11,I12)=I23
    GO   TO   490
690 J(I11,I12)=I24
    GO   TO   490
700 J(I11,I12)=I25
    GO   TO   490
710 J(I11,I12)=I26
490 IF   (J(I11,I12).LE.4)   GO   TO   500
    I(I11,I12)=J(I11,I12)
    GO   TO   111
500 CONTINUE
    DO   125   M3=1,N
    DO   125   M4=1,N
    J1(1,M3,M4)=J(M3,M4)
125 CONTINUE
    DO   82   K1=1,N
    DO   82   K2=1,N
    DO   82   K3=1,N
    IF   (J(J(K1,K2),K3).NE.J(K1,J(K2,K3)))   GO   TO   94
 82 CONTINUE
    IF   (H.EQ.1)   GO   TO   123
    DO   124   M1=1,N
    DO   124   M2=1,N
    IF   (J1(2,M1,M2).NE.J1(1,M1,M2))   GO   TO   123
124 CONTINUE
    GO   TO   94
123 WRITE   (6,91)   H,((J(L,K),K=1,4),L=1,4)
    WRITE   (7,91)   H,((J(L,K),K=1,4),L=1,4)
    WRITE   (6,92)   M
 91 FORMAT   (I15,16I2)
 92 FORMAT   (I60)
 94 H=H+1
    DO   126   M5=1,N
    DO   126   M6=1,N
    J1(2,M5,M6)=J1(1,M5,M6)
126 CONTINUE
120 CONTINUE
 10 CONTINUE
    STOP
    END
```

GENERATION OF ORDER FOUR SEMIGROUPS PART THREE

```
      INTEGER  H
      DIMENSION  I(20,20),M(900,5,5),LP(5,5),L(5,5),LF(5),LR
      N=4
      M9=0
      M8=0
      DO  100  K9=1,900
      L4=1
      N4=1
      L5=0
      N5=0
      J6=1
   15 READ  (5,20)  H,((I(J,K),K=1,4),J=1,4)
      IF  (H.EQ.9999999)  GO  TO  105
   20 FORMAT  (I15,16I2)
      DO  999  KK=1,N
      DO  999  LL=1,N
      DO  999  MM=1,N
      IF  (I(I(KK,LL),MM).NE.I(KK,I(LL,MM)))  GO  TO  998
  999 CONTINUE
      GO  TO  996
  998 M20=M9+1
      WRITE  (6,997)  M20
  997 FORMAT  (30X'NO',I9)
      GO  TO  100
  996 CONTINUE
      M9=M9+1
      M8=M8+1
      WRITE  (6,21)  M9,((I(J,K),K=1,4),J=1,4)
      WRITE  (7,210)  ((I(J,K),K=1,4),J=1,4)
  210 FORMAT (///2X,I2,4X,I2,4X,I2,4X,I2,//2X,I2,4X,I2,4X,I2
     1I2,4X,I2,4X,I2,4X,I2,//2X,I2,4X,I2,4X,I2,4X,I2)
   21 FORMAT  (///1X'PERMUTATION',I9,'  IS  A  SEMIGROUP',//
     14X,I2,4X,I2,//2X,I2,4X,I2,4X,I2,4X,I2,//2X,I2,4X,I2,4X
     22X,I2,4X,I2,4X,I2,4X,I2)
      DO  50  I30=1,N
      DO  50  J10=1,N
   50 M(M8,I30,J10)=I(I30,J10)
      WRITE  (6,51)  M8
      WRITE  (7,51)  M8
   51 FORMAT  (10X'SEMIGROUP  NUMBER',I7)
      IF  (M9.EQ.1)  GO  TO  5
      DO  90  I40=1,N
      DO  90  I41=1,N
      DO  90  I42=1,N
      DO  90  I43=1,N
      LF(1)=I40
      LF(2)=I41
      LF(3)=I42
      LF(4)=I43
      LR(I40)=1
      LR(I41)=2
      LR(I42)=3
      LR(I43)=4
      IF  ((I41.EQ.I42).OR.(I41.EQ.I43).OR.(I42.EQ.I43))  GO
      IF  ((I40.EQ.I41).OR.(I40.EQ.I42).OR.(I40.EQ.I43))  GO
      DO  80  K10=1,N
      DO  80  M10=1,N
      DO  80  N10=1,N
      IF  (I(M10,N10).EQ.K10)  L(M10,N10)=LF(K10)
   80 CONTINUE
      LP(1,1)=L(LR(1),LR(1))
      LP(1,2)=L(LR(1),LR(2))
      LP(1,3)=L(LR(1),LR(3))
      LP(1,4)=L(LR(1),LR(4))
```

```
      LP(2,1)=L(LR(2),LR(1))
      LP(2,2)=L(LR(2),LR(2))
      LP(2,3)=L(LR(2),LR(3))
      LP(2,4)=L(LR(2),LR(4))
      LP(3,1)=L(LR(3),LR(1))
      LP(3,2)=L(LR(3),LR(2))
      LP(3,3)=L(LR(3),LR(3))
      LP(3,4)=L(LR(3),LR(4))
      LP(4,1)=L(LR(4),LR(1))
      LP(4,2)=L(LR(4),LR(2))
      LP(4,3)=L(LR(4),LR(3))
      LP(4,4)=L(LR(4),LR(4))
   79 L20=1
   59 DO  60  I31=1,N
      DO  60  J11=1,N
      IF  (LP(I31,J11).NE.M(L20,I31,J11))  GO  TO  61
   60 CONTINUE
      GO  TO  65
   61 L20=L20+1
      IF  (L20.EQ.M8)  GO  TO  90
      GO  TO  59
   65 WRITE  (6,66)  L20
   66 FORMAT  (10X'SEMIGROUP  IS  ISOMORPHIC  TO  SEMIGROUP
      M8=M8-1
      GO  TO  100
   90 CONTINUE
    5 CONTINUE
      DO  500  I500=1,N
      DO  500  J500=1,N
      K500=N+1-I500
      L500=N+1-J500
      N501=1
      N502=2
      N503=3
      N504=4
      IF  (I500.EQ.J500)  GO  TO  501
      DO  502  I502=1,N
      DO  502  J502=1,N
      IF  ((I502.EQ.K500).OR.(I502.EQ.L500).OR.(J502.EQ.K500
     1).OR.(J502.EQ.L500))  GO  TO  502
      IF  ((I(I502,J502).EQ.K500).OR.(I(I502,J502).EQ.L500))
     1  GO  TO  500
  502 CONTINUE
      M500=K500+L500
      IF  (M500.EQ.3)  GO  TO  510
      IF  (M500.EQ.4)  GO  TO  511
      IF  (M500.EQ.5)  GO  TO  512
      IF  (M500.EQ.6)  GO  TO  513
      IF  (M500.EQ.7)  GO  TO  514
  510 WRITE  (6,503)  N503,N504
      GO  TO  500
  511 WRITE  (6,503)  N502,N504
      GO  TO  500
  512 IF  ((K500.EQ.1).OR.(L500.EQ.1))  GO  TO  515
      WRITE  (6,503)  N501,N504
      GO  TO  500
  515 WRITE  (6,503)  N502,N503
      GO  TO  500
  513 WRITE  (6,503)  N501,N503
      GO  TO  500
  514 WRITE  (6,503)  N501,N502
  503 FORMAT  (10X'SEMIGROUP  HAS  A  SUBSEMIGROUP  OF
     1ORDER  TWO  ',2I2)
      GO  TO  500
  501 DO  504  I504=1,N
      DO  504  J504=1,N
      IF  ((I504.EQ.K500).OR.(J504.EQ.K500))  GO  TO  504
      IF  (I(I504,J504).EQ.K500)  GO  TO  500
  504 CONTINUE
      IF  (K500.EQ.1)  GO  TO  506
      IF  (K500.EQ.2)  GO  TO  507
      IF  (K500.EQ.3)  GO  TO  508
```

```
      IF  (K500.EQ.4)  GO  TO  509
  506 WRITE  (6,505)  N502,N503,N504
      GO  TO  500
  507 WRITE  (6,505)  N501,N503,N504
      GO  TO  500
  508 WRITE  (6,505)  N501,N502,N504
      GO  TO  500
  509 WRITE  (6,505)  N501,N502,N503
  505 FORMAT  (10X'SEMIGROUP  HAS  A  SUBSEMIGROUP  OF
     1ORDER  THREE  ',3I2)
  500 CONTINUE
      DO  22  K1=1,N
      DO  22  K2=1,N
      IF  (I(K1,K2).NE.I(K2,K1))  GO  TO  26
   22 CONTINUE
      WRITE  (7,23)
      WRITE  (6,23)
   23 FORMAT  (10X'SEMIGROUP  IS  COMMUTATIVE')
   26 DO  27  K4=1,N
      IF  (I(L4,K4).NE.K4)  GO  TO  29
   27 CONTINUE
      WRITE  (7,28)  L4
      WRITE  (6,28)  L4
      L5=L4
   28 FORMAT  (10X'SEMIGROUP  HAS  LEFT  IDENTITY  X =',I2,/
     1 THAT  XY=Y')
   29 L4=L4+1
      IF  (L4.GT.N)  GO  TO  30
      GO  TO  26
   30 DO  31  M6=1,N
      IF  (I(M6,N4).NE.M6)  GO  TO  33
   31 CONTINUE
      WRITE  (7,32)  N4
      WRITE  (6,32)  N4
      N5=N4
   32 FORMAT  (10X'SEMIGROUP  HAS  RIGHT  IDENTITY  Z =',I2,
     1  THAT  YZ=Y')
   33 N4=N4+1
      IF  (N4.GT.N)  GO  TO  34
      GO  TO  30
   34 IF  ((N5.NE.L5).OR.((N5.EQ.0).AND.(L5.EQ.0)))  GO  TO
      WRITE  (7,35)
      WRITE  (6,35)
   35 FORMAT  (10X'SEMIGROUP  HAS  IDENTITY')
   36 DO  37  J5=1,N
      IF  (I(J6,J5).EQ.N5)  GO  TO  38
   37 CONTINUE
      GO  TO  24
   38 J6=J6+1
      IF  (J6.GT.N)  GO  TO  39
      GO  TO  36
   39 WRITE  (6,40)
      WRITE  (7,40)
   40 FORMAT  (10X'SEMIGROUP  IS  A  GROUP')
   24 CONTINUE
  100 CONTINUE
  105 CONTINUE
      STOP
      END
```

BIBLIOGRAPHY

Clifford, A. H., and Preston, G. B., The Algebraic Theory of Semigroups, v.1, American Mathematical Society, 1961.

Ljapin, E. S., Semigroups, American Mathematical Society, 1963.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	20
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Naval Ship Systems Command, Code 2052 Department of the Navy Washington, D. C. 20370	1
4.	Professor C. O. Wilde, Code 56Wm Department of Mathematics Naval Postgraduate School Monterey, California 93940	1
5.	LTJG James S. Cullen, USN 4914 Ravensworth Road Annandale, Virginia 22003	2

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*

Naval Postgraduate School
Monterey, California 93940

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

An Algorithm for Computing Non-Isomorphic Semigroups of Finite Order

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Master's Thesis; June 1969

5. AUTHOR(S) *(First name, middle initial, last name)*

James Stephen Cullen

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1969	79	2

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

~~Distribution of this document is unlimited.~~

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Naval Postgraduate School
Monterey, California 93940

13. ABSTRACT

In this paper an algorithm for computing semigroups of finite order is discussed. A computation procedure is developed to generate, for any specified finite order, all semigroups which are distinct up to isomorphism. Additional restrictions are also placed in the generating procedure to produce all groups of the given finite order. The algorithm was placed on the computer and the numerical results for orders one through four obtained.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Security Classification

A-31408

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Algorithm for computing semigroups of finite order						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821